Functions

Summary Lecture

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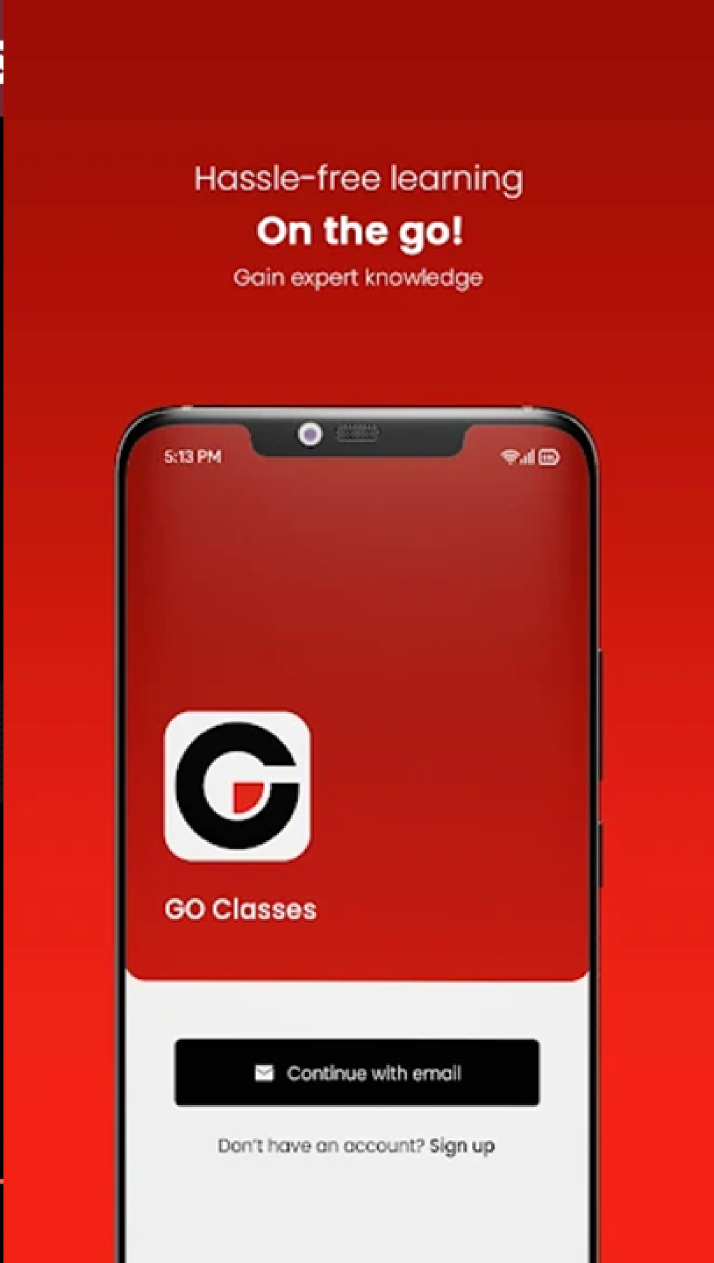

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Functions

Complete Summary

Functions

Objectives:

1. Determine whether a relationship is a function or not
2. Determine the domain, co-domain, range of a function, and the inverse image of x
3. Prove or disprove whether a function is one-to-one or not
4. Determine whether a function is onto or not
5. Determine the inverse of a one-to-one correspondence
6. Determine the composition of two functions
7. Show that if two functions are one-to-one (onto) so too is their composition

Function Definition

function f : set $A \longrightarrow$ set B

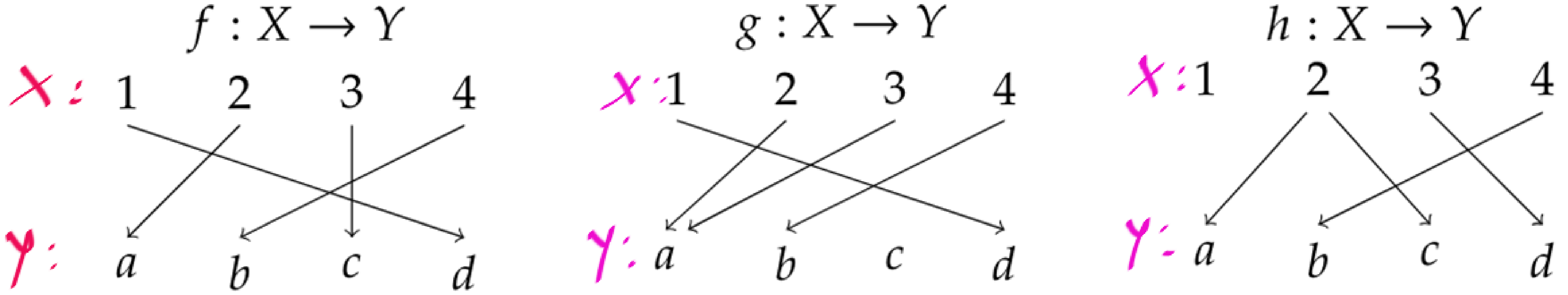
Such that f maps EVERY element of set A (individually) to exactly one element of set B.

A, B may be Same or Different

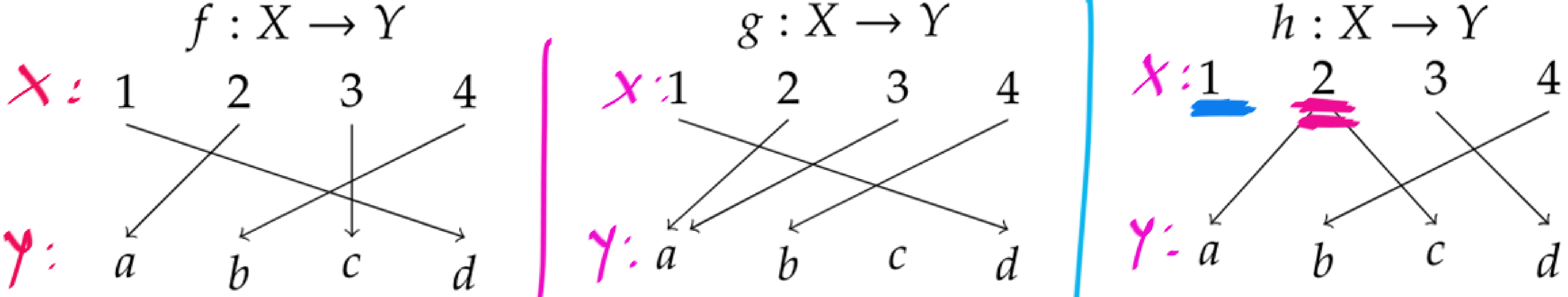
Function f: A → B

Function from Set A to Set B
is a structure which maps EVERY
element of A (individually) to EXACTLY
ONE element of B.

Which of the following diagrams represent a function? Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$.



Which of the following diagrams represent a function? Let $X = \{1,2,3,4\}$ and $Y = \{a,b,c,d\}$.



$f : X \to Y$

f: a function

$g : X \to Y$

g: a function

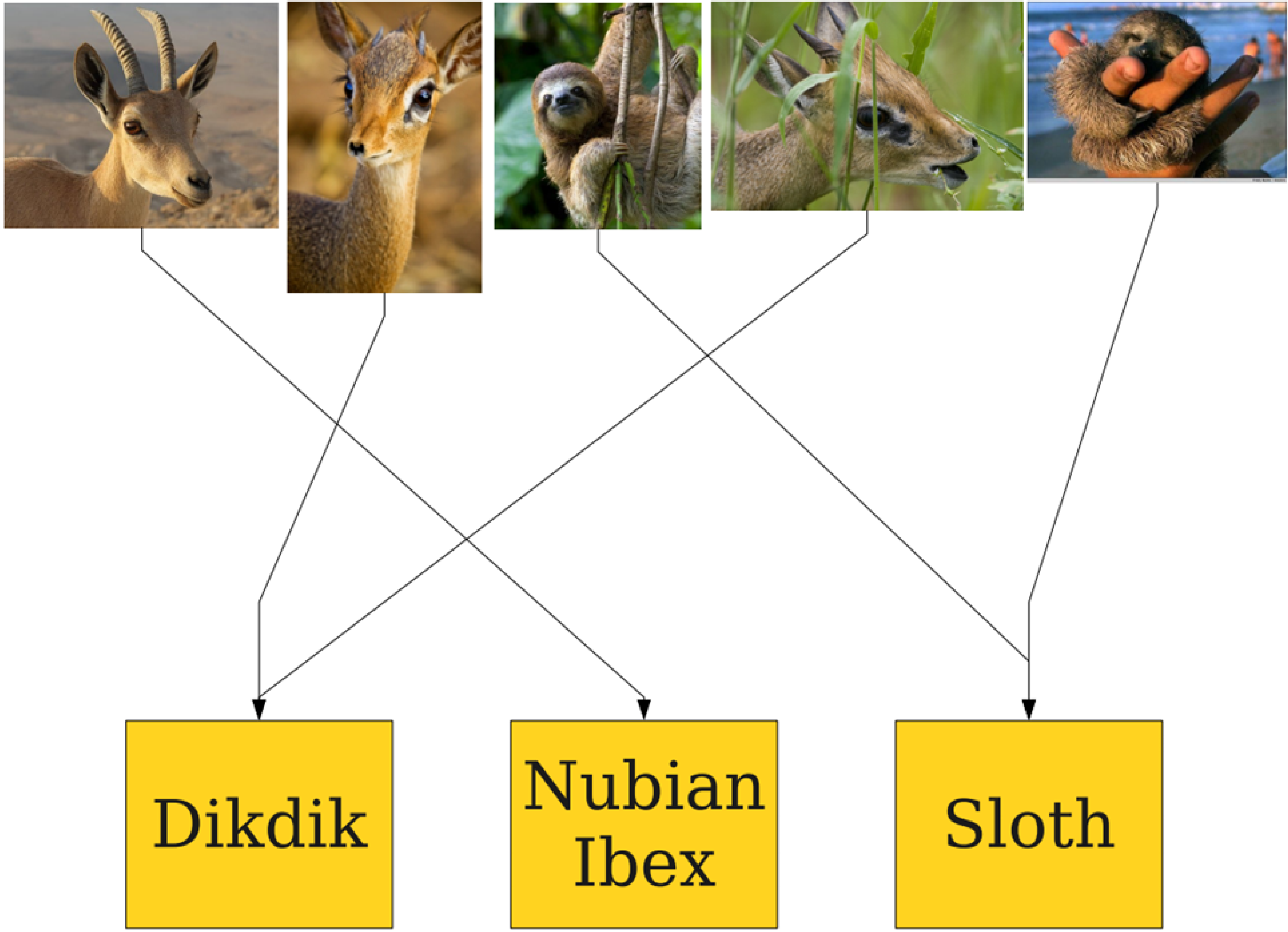
$h : X \to Y$

h: NOT a function

Functions

- A **function** f is a mapping such that every element of A is associated with a single element of B.
- If f is a function from A to B, then
  - we call A the **domain** of f.
  - we call B the **codomain** of f.
- We denote that f is a function from A to B by writing

$$f : \underbrace{A}_{\text{Domain}} \rightarrow \underbrace{B}_{\text{Co-Domain}}$$

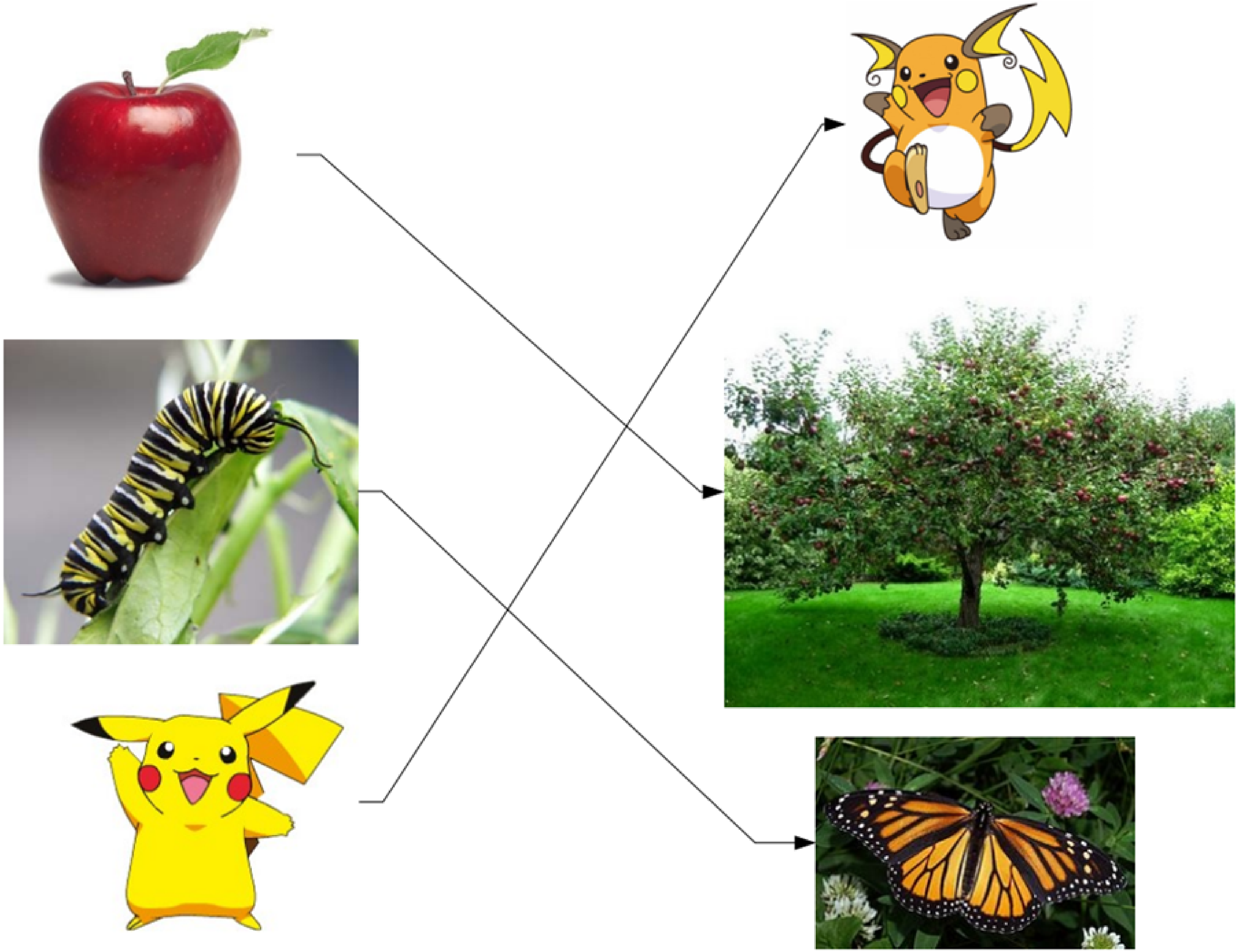


Dikdik

Nubian Ibex

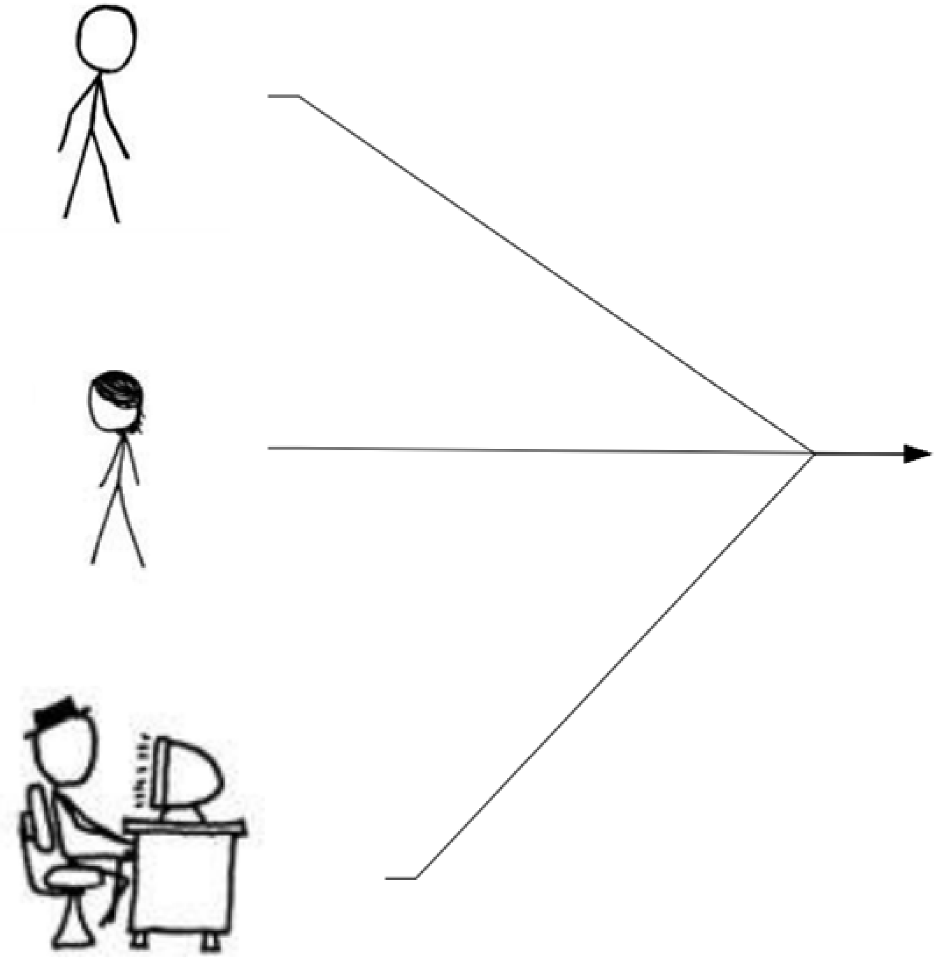
Sloth

a function



a

function

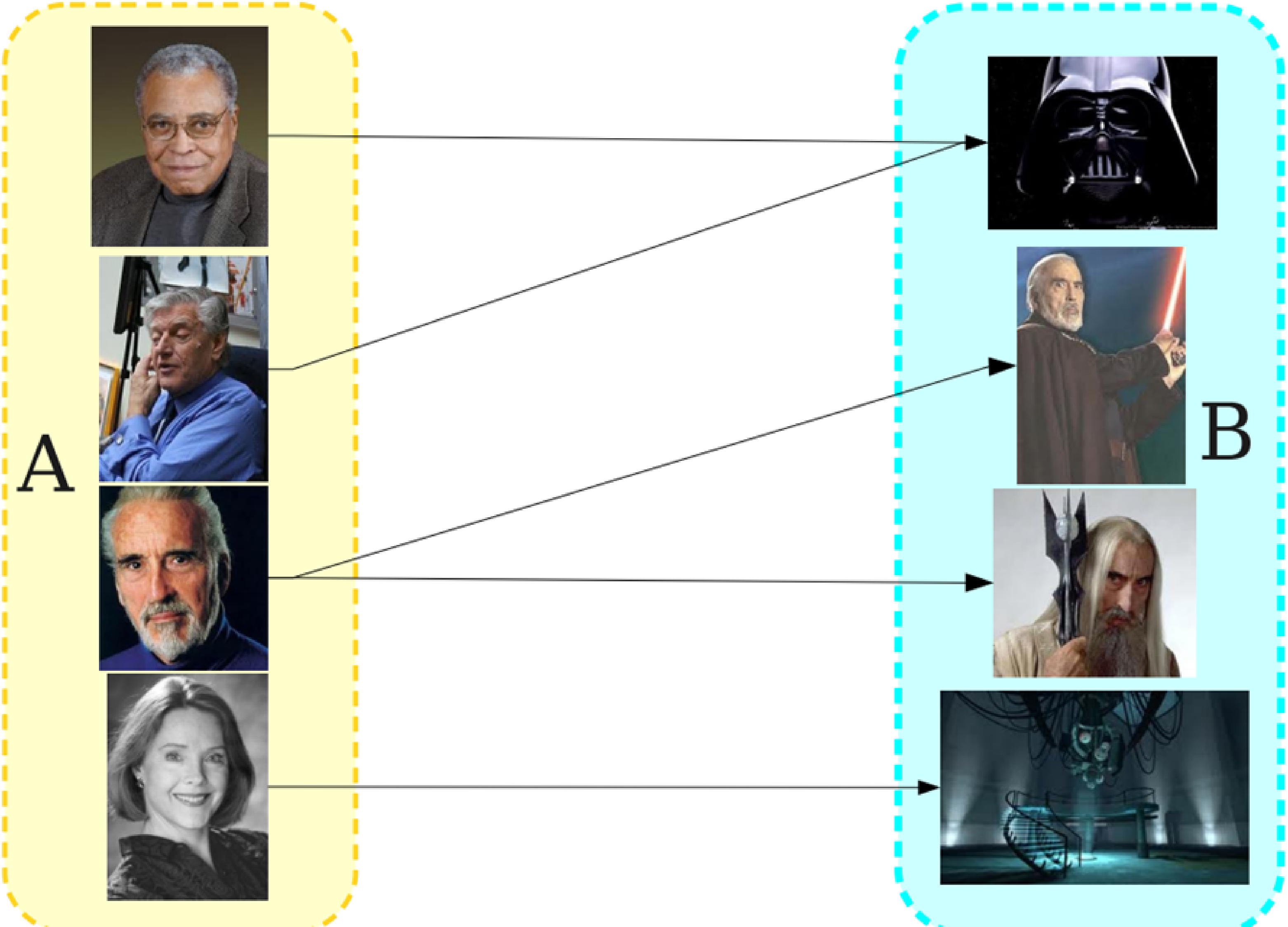


Black and White

a ✓

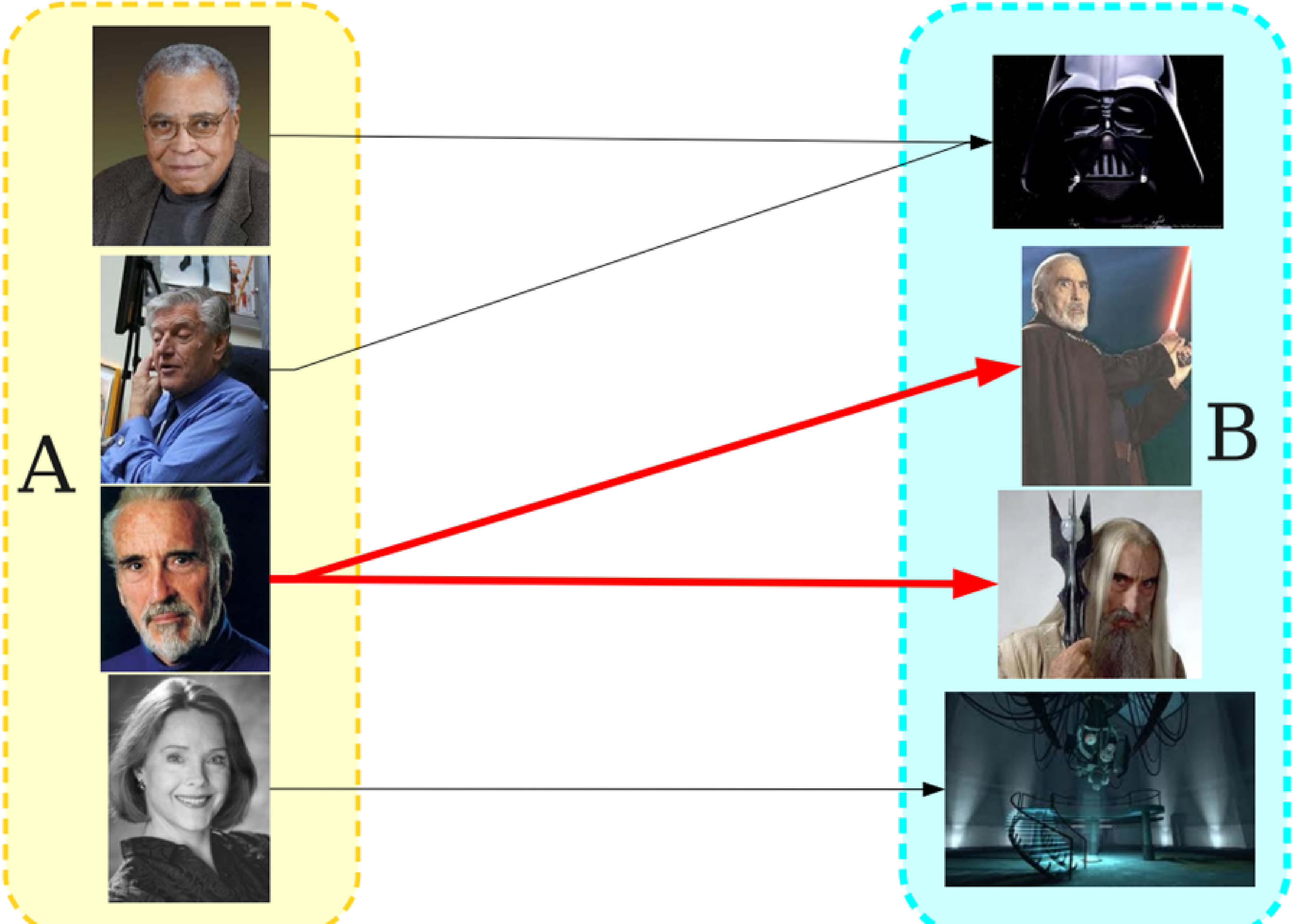
function

Is This a Function from A to B?

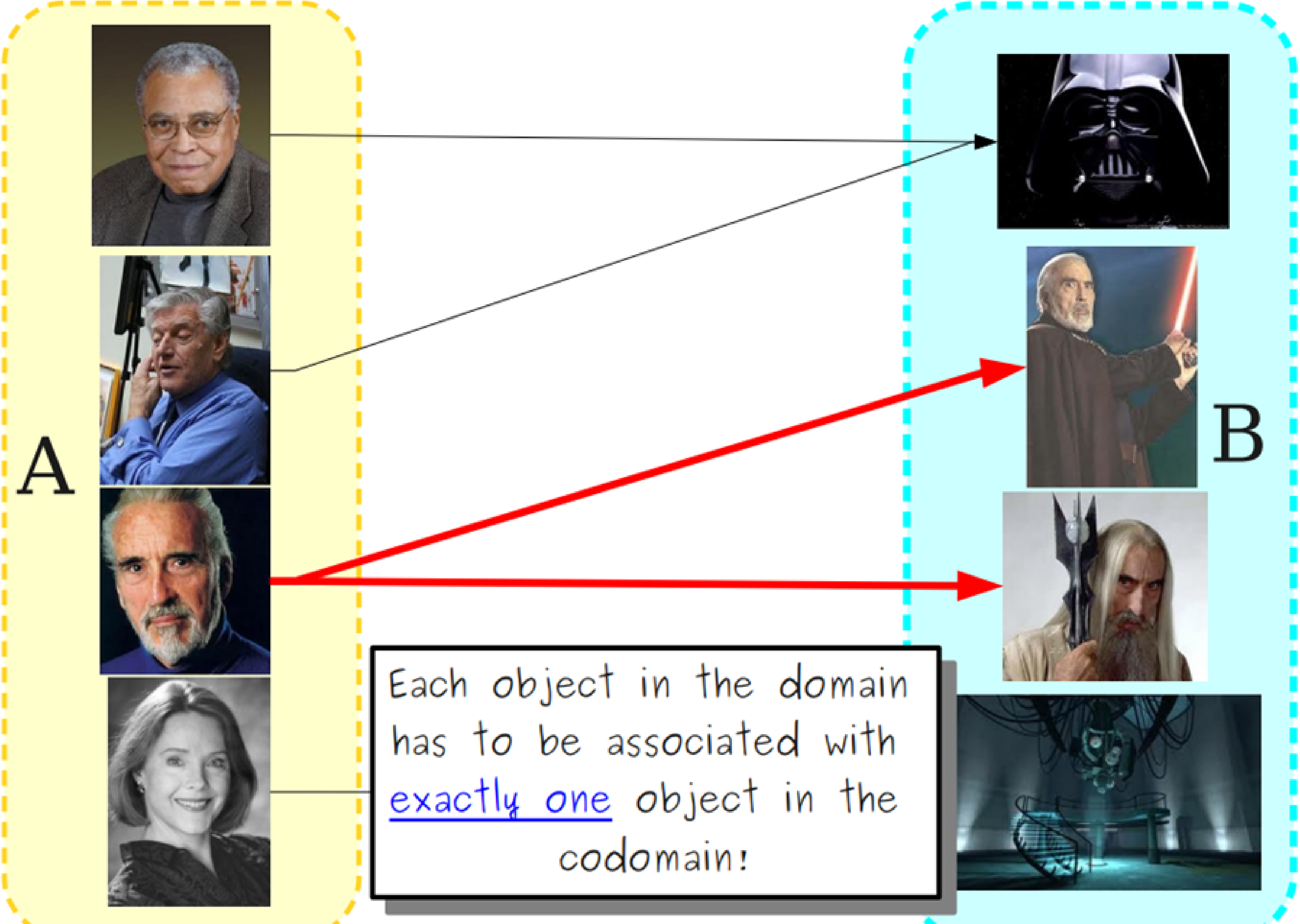


Not a function

Is This a Function from A to B?

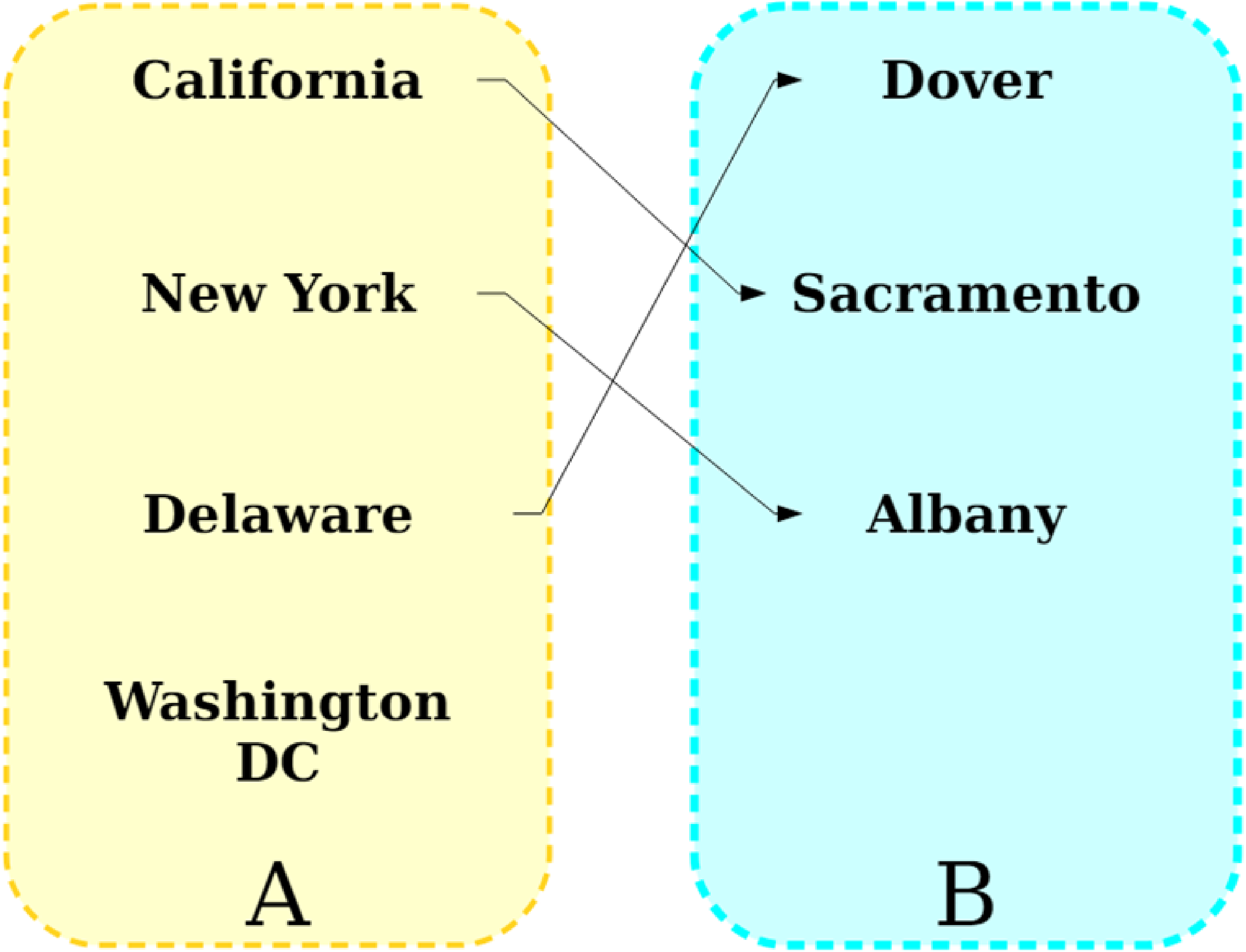


Is This a Function from A to B?



Each object in the domain has to be associated with exactly one object in the codomain!

Is this a function from A to B?



California

New York

Delaware

Washington DC

A

Dover

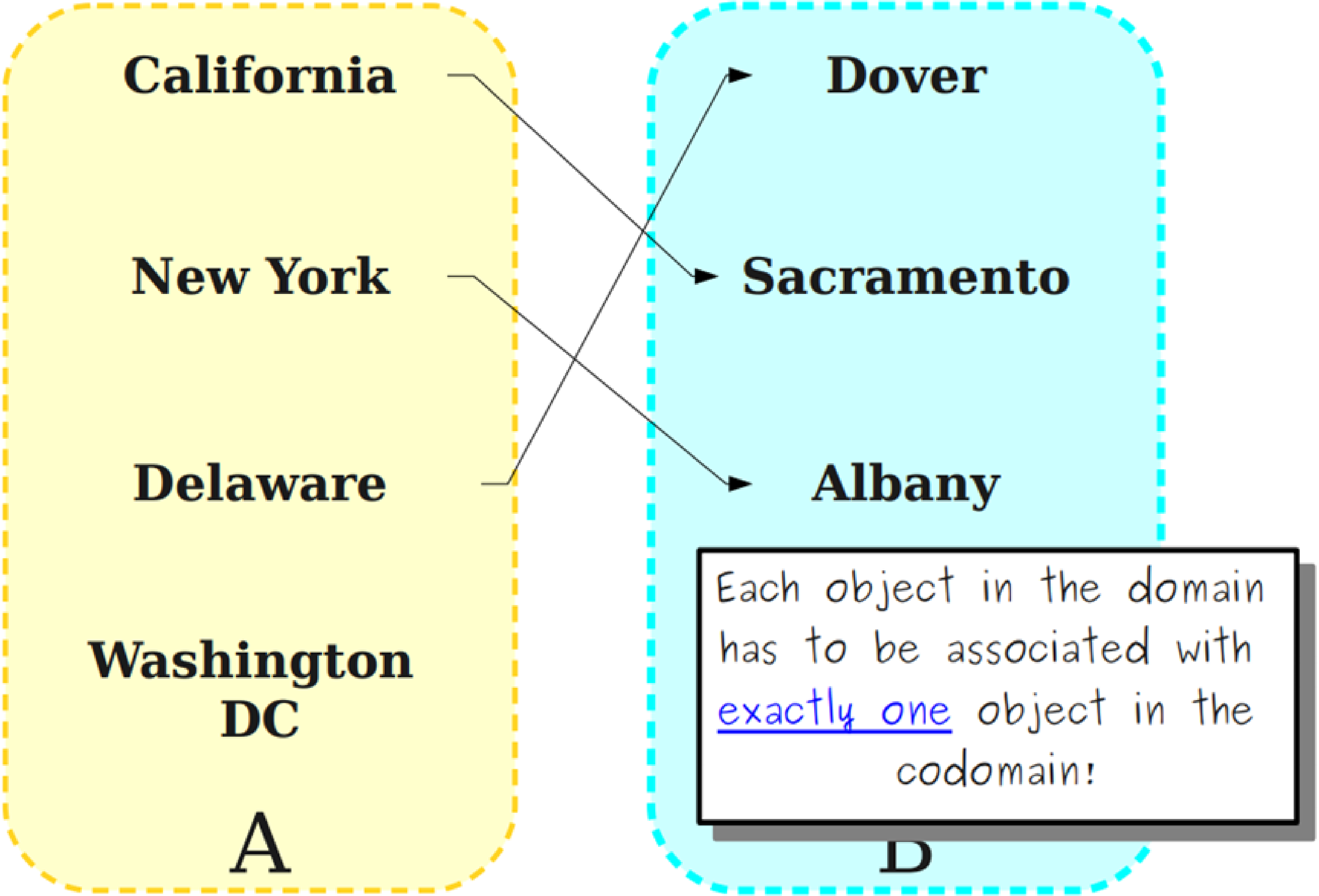
Sacramento

Albany

B

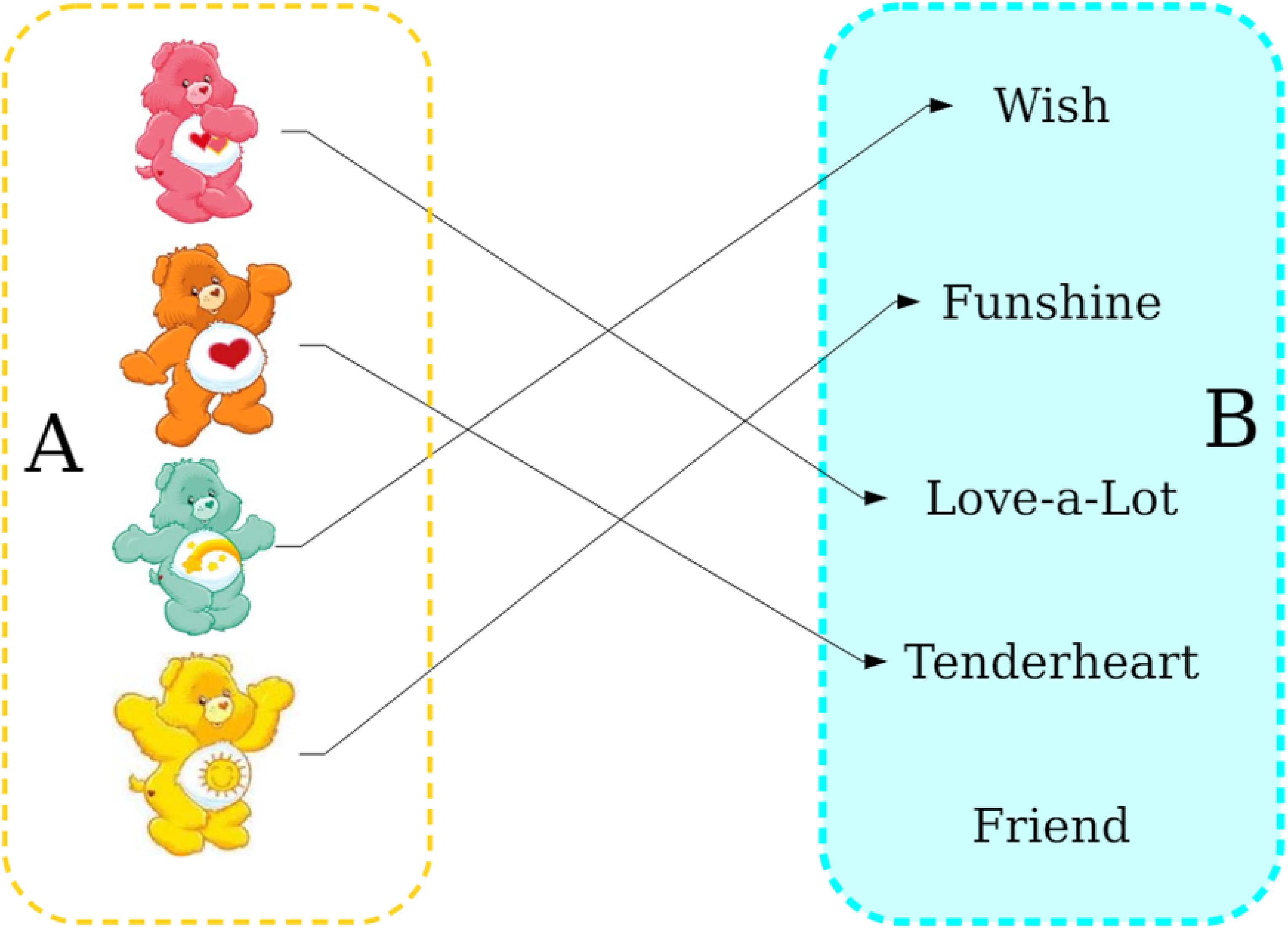
NOT a function

Is This a Function from A to B?



Each object in the domain has to be associated with exactly one object in the codomain!

Is this a function from A to B?



A

Wish

Funshine

Love-a-Lot

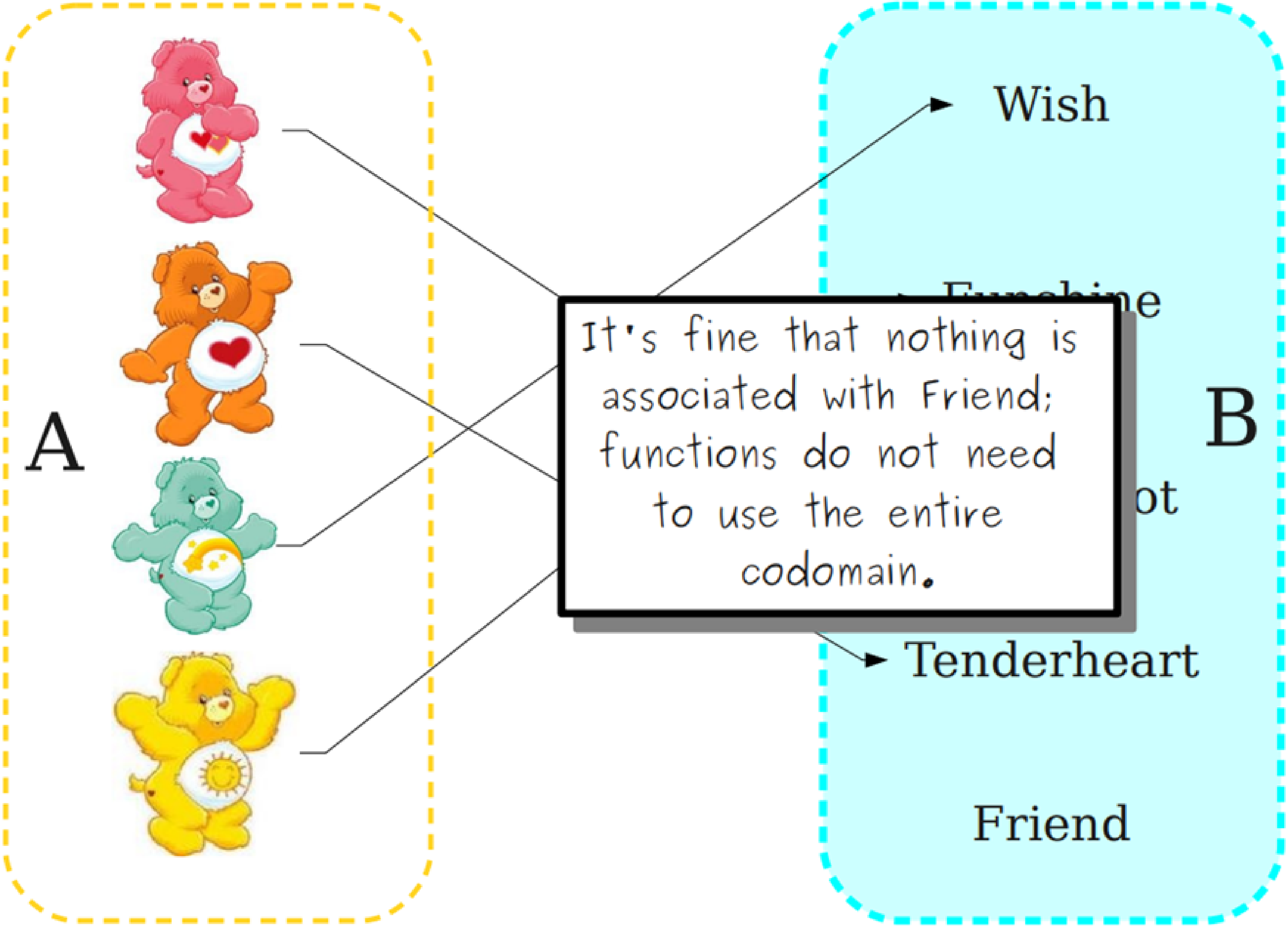
Tenderheart

Friend

B

✓ a function

Is This a Function from A to B?



It's fine that nothing is associated with Friend; functions do not need to use the entire codomain.

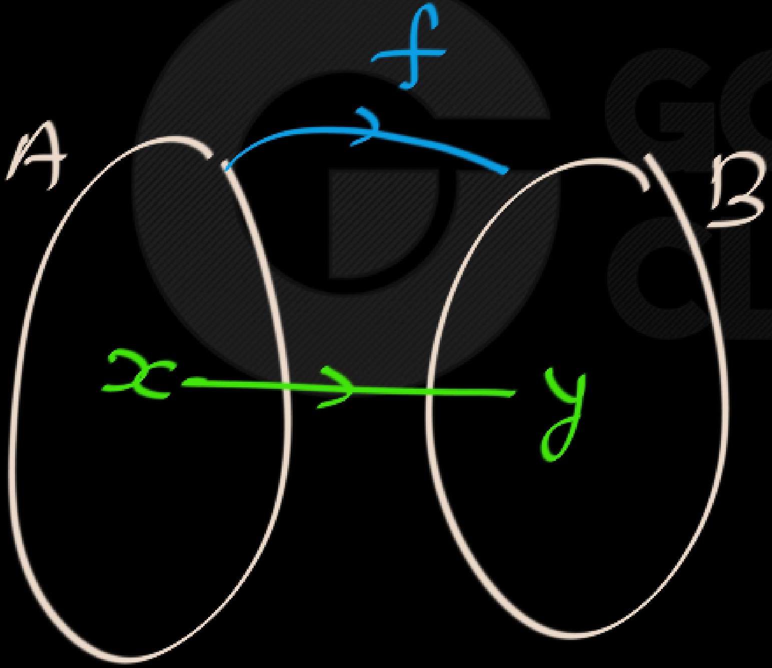
Function Terminology

$f : A \to B$

function

Domain

Co-Domain



x is mapped to y.

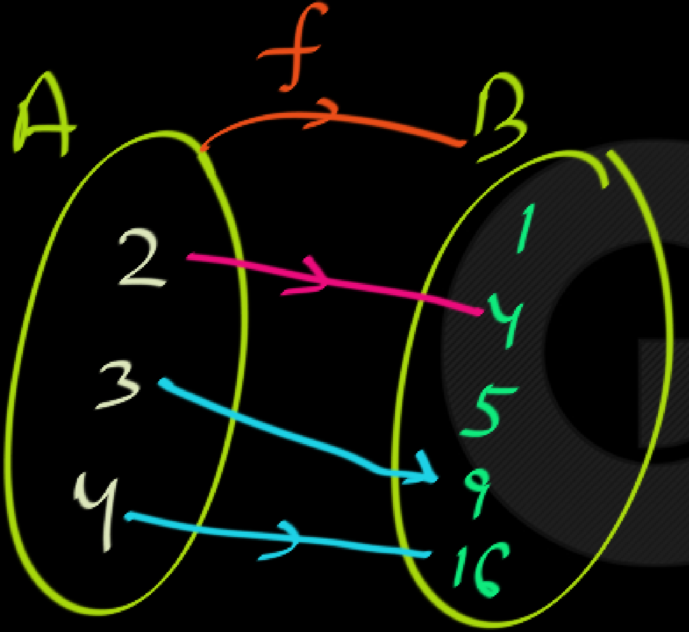
Image of x is y.

f-value of x is y.

$f(x) = y$

x is pre-image of y.

function ≡ mapping ≡ Transformation



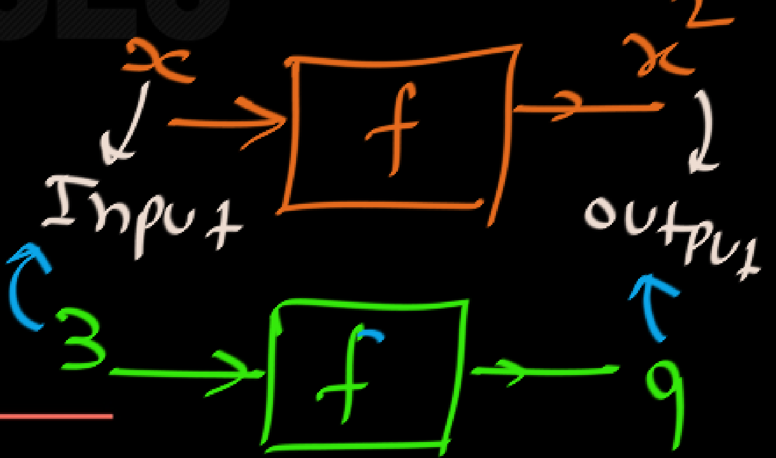
$\boxed{f(x) = x^2}$ means

Image of x is x^2.

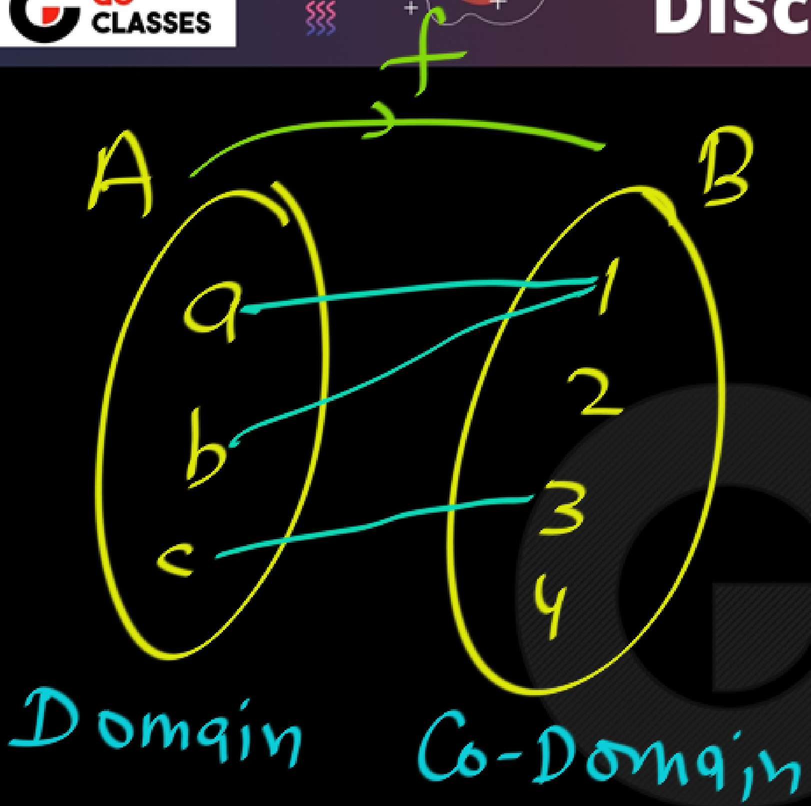
$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f(4) = 4^2 = 16$

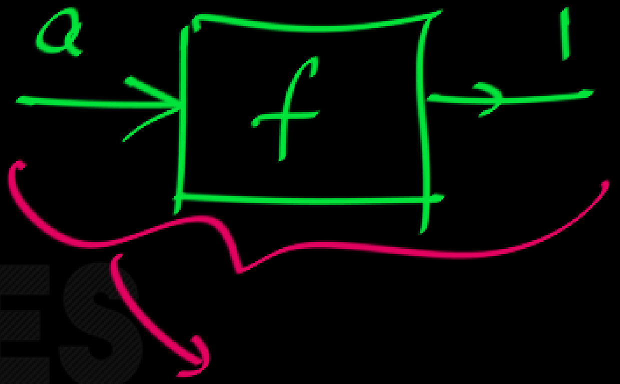


$\boxed{f(x) = x^2}$



"Image of a" = 1

$f(a) = 1$

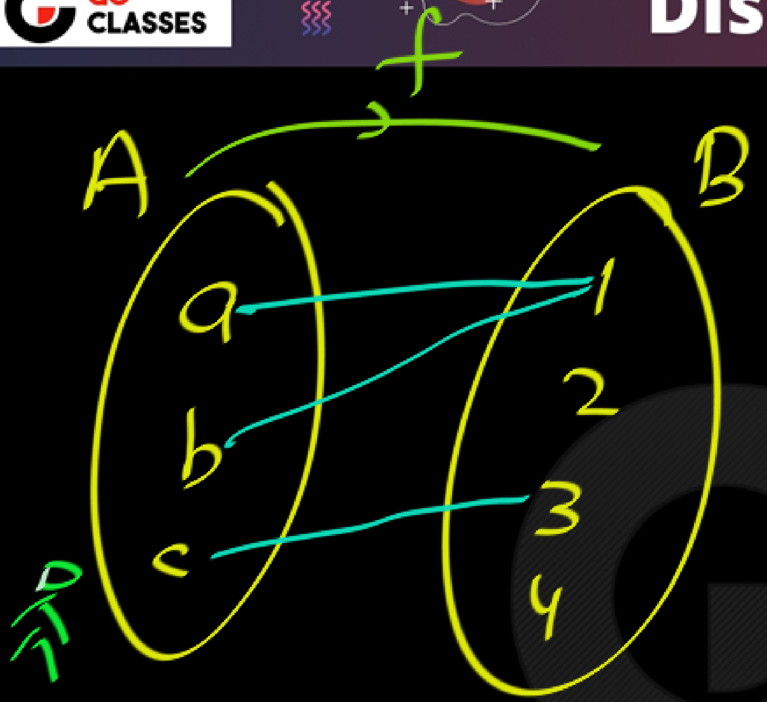


$f(\underset{\text{input}}{a}) = \underset{\text{output}}{1}$

Pre-image of 1 : {a, b}

No Pre-image of 2 in f.

$$\underbrace{\text{Pre-image}} \equiv \underbrace{\text{inverse image}}$$



Range(f) = { 1, 3 }

Reachable elements of Co-Domain

Domain(f) = A = {a, b, c}

Co-Domain (f) = B = {1, 2, 3, 4}

Range(f) ⊆ Co-Domain (f)

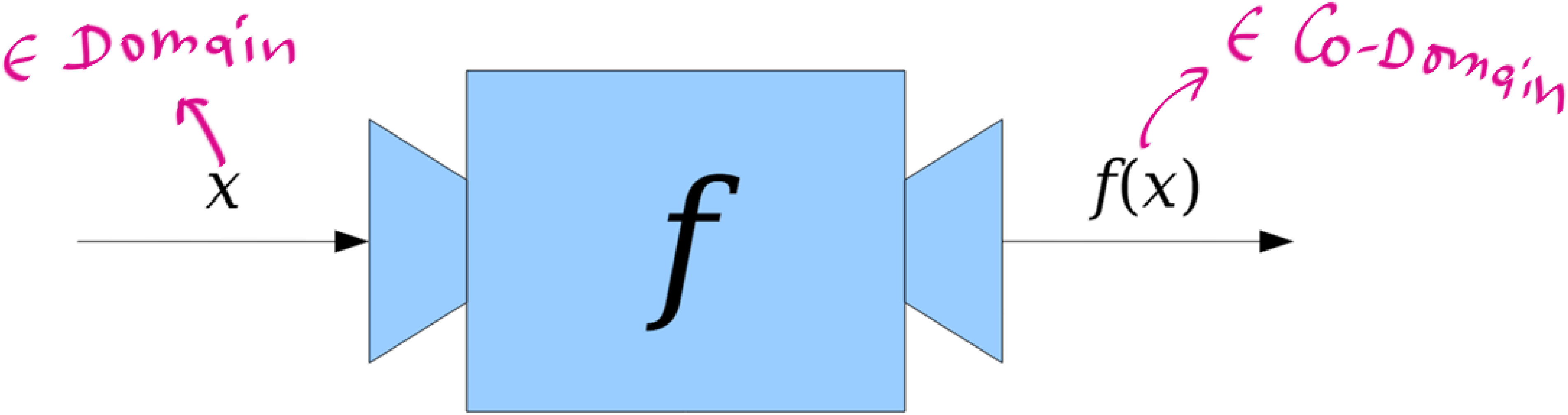
function $f : A \longrightarrow B$

Range $(f) = \{ y \mid y \in B$; y has at least one pre-image $\}$

$$\text{Range}(f) = \{ y \in B \mid \exists x \in A, f(x) = y \}$$

Remark: Functions are sometimes also called **mappings** or **transformations**.

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



Definitions

Let A and B be sets.

- A **function** f from A to B is an assigment of exactly one element of B to each element of A. *Note: exactly one element of* B. We can write this as $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. We write $f : A \rightarrow B$.

  > Given an input element a in A, there is a unique output element b in B that is related to a by f.

- A is the **domain** of f and B is the **co-domain** of f.
- If $f(a) = b$, we say that b is the **image** of a and a is a **pre-image**, or **inverse image**, of b.
- The **range** of f is the set of all images of elements of A.

We say that a function “maps” one set to another.

Function Requirements

There are rules for functions to be well defined, or correct.

- No element of the domain must be left unmapped.
- No element of the domain may map to more than one element of the co-domain.

It is allowable, however, for elements of the co-domain to be unmapped or to have multiple elements from the domain map to single elements in the co-domain.

So basically what we are saying is that every element in the domain has to be mapped to an output, and there can be only one output for every input. No input can have to possible outputs.

1. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(n) = 3n$. The domain and codomain are both the set of integers. However, the range is only the set of integer multiples of 3.

$$\left.\begin{aligned} &f : Z \rightarrow Z \\ &f(n) = 3n \end{aligned}\right\} \rightarrow \underline{\text{a function}}$$

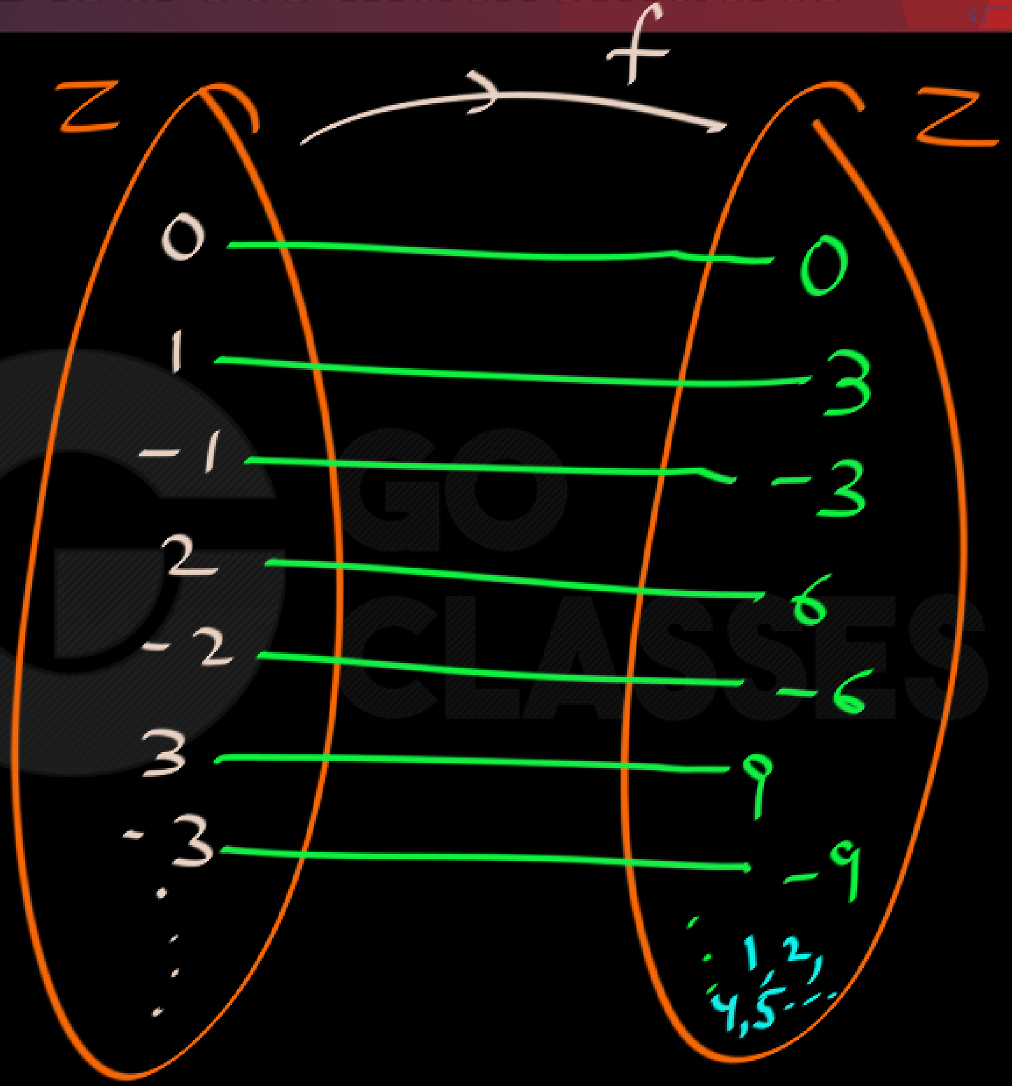
Z = Integers = $\{0, +1, -1, +2, -3, ---\}$

$$\text{Range}(f) = \{m \mid m \text{ is multiple of } 3\}$$

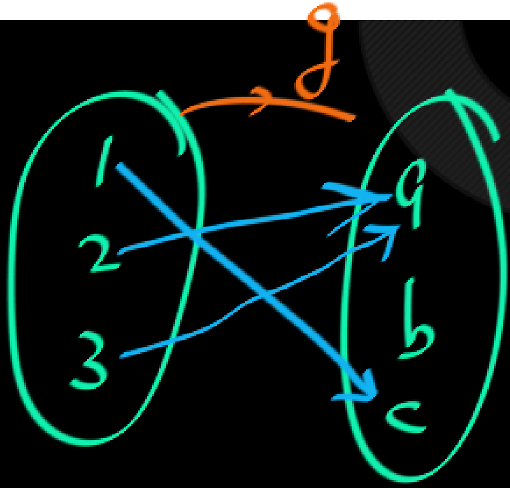
$f : Z \to Z$

$f(n) = 3n$

$n \to \boxed{f} \to 3n$



2. $g : \{1, 2, 3\} \rightarrow \{a, b, c\}$ defined by $g(1) = c$, $g(2) = a$ and $g(3) = a$. The domain is the set $\{1, 2, 3\}$, the codomain is the set $\{a, b, c\}$ and the range is the set $\{a, c\}$. Note that $g(2)$ and $g(3)$ are the same element of the codomain. This is okay since each element in the domain still has only one output.



$g(1) = c \; ; \; g(2) = a \; ; \; g(3) = a$

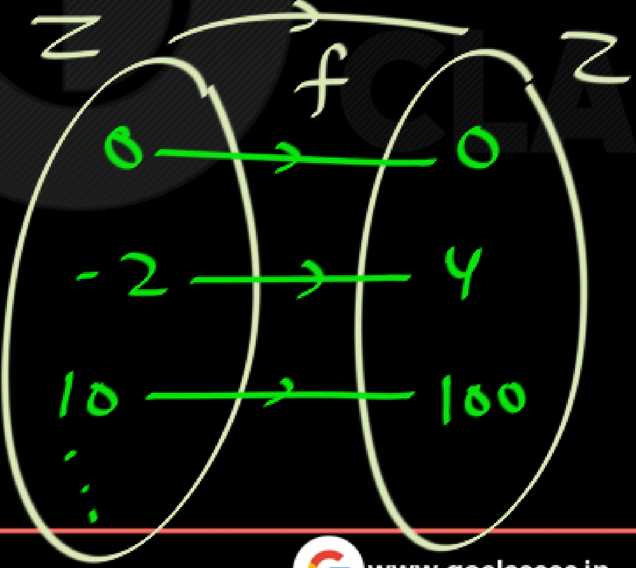
$\text{Range}(g) = \{a, c\}$

No pre-image for b.

Let $f: \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$. ◀

$$\left.\begin{array}{l} f: Z \to \underline{Z} \\ f(x) = x^2 \end{array}\right\} \to \text{function? } \underline{\text{Yes.}}$$

$\underbrace{\text{Image of } 2}$: 4



Range(f) :

$$\left\{0, 1, 2^2, 3^2, 4^2, 5^2, \ldots\right\}$$

Range ≠ Co-Domain

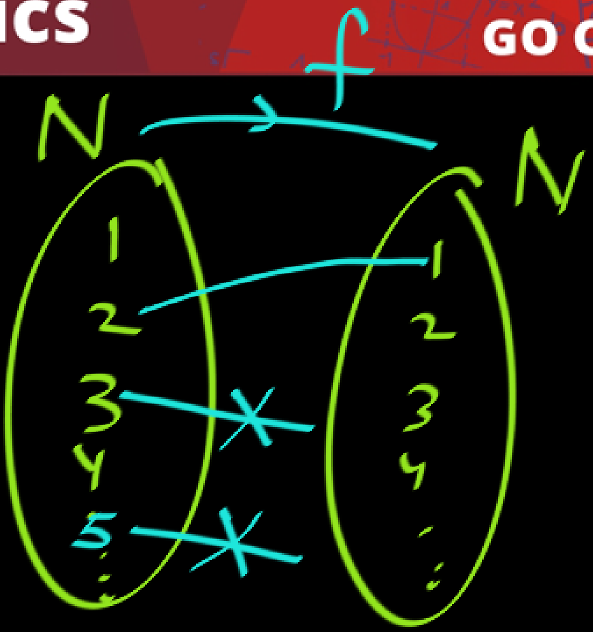
Pre-image of 2 : $\underline{\text{None}}$

$$\underbrace{f : N \to \underline{N} \; ; \; f(n) = \frac{n}{2}}$$

↓

a function ? **No**

Image of 3 : None



Natural Numbers

$N = \{1, 2, 3, 4, - - -\}$

1. $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = \frac{n}{2}$. The reason this is not a function is because not every input has an output. Where does f send 3? The rule says that $f(3) = \frac{3}{2}$, but $\frac{3}{2}$ is not an element of the codomain.

2. Determine whether f is a function from **Z** to **R** if

Integers Real

a) $f(n) = \pm n.$

b) $f(n) = \sqrt{n^2 + 1}.$

c) $f(n) = 1/(n^2 - 4).$

2. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if

✗ a) $f(n) = \pm n$.

✓ b) $f(n) = \sqrt{n^2 + 1}$.

✗ c) $f(n) = 1/(n^2 - 4)$.

$\underbrace{f(2) = 2, -2}$

Co-Domain

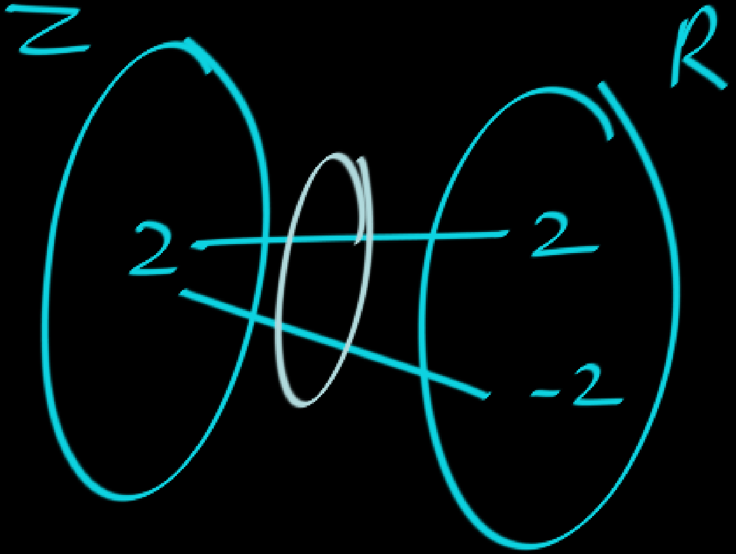
$\infty \notin R$

Image of 2 : $\frac{1}{0} \notin R$

None

ⓐ $f : Z \to R$

$$\left.\begin{array}{l} f : Z \to R \\ f(n) = \pm n \end{array}\right\} \underline{\text{Not a function}}$$



ⓑ $f : Z \to R$

$f(n) = \sqrt{n^2+1}$

$f(0) = \sqrt{1} = 1 \in R$

$f(2) = \sqrt{5} \in R$ $f(-2) = \underline{\sqrt{5} \in R}$

$\sqrt{9} = 3$ ✓

$\sqrt{9} = \pm 3$ ✗

$\sqrt{16} \neq \pm 4$

$\sqrt{16} = 4$ ✓

$$\underbrace{x^2 = 9} \Longrightarrow x = \pm 3$$

$$x^2 = 9 \Longrightarrow x = \pm \overset{3}{\boxed{\sqrt{9}}} = \pm 3$$

$\frac{1}{0} \notin R$; $\frac{1}{0} = \infty \notin R$

$\infty \notin R$

∞ : NoT a Number

Function Equality

Function Equality:

When can we say that Two functions are SAME ?

$f : \{1,2\} \rightarrow N \; ; \; f(x) = x^2$

$g : \{1,2\} \rightarrow Z \; ; \; g(x) = x^2$

Same or Different ?

Function Equality:

When can we say that Two functions are SAME ?

$f : \{1,2\} \rightarrow \underline{\underline{N}}$; $f(x) = x^2$

$g : \{1,2\} \rightarrow \underline{\underline{Z}}$; $g(x) = x^2$

Same or Different ✓

Co-Domain Different

function $f : A \to B$; function $g : C \to D$

$f = g$ iff

✓

① Domain(f) = Domain(g)

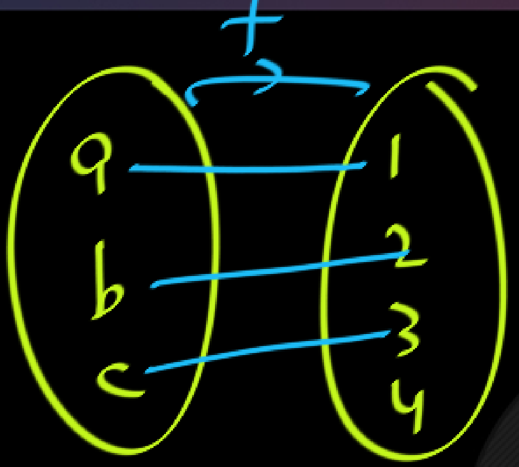
i.e. $A = C$

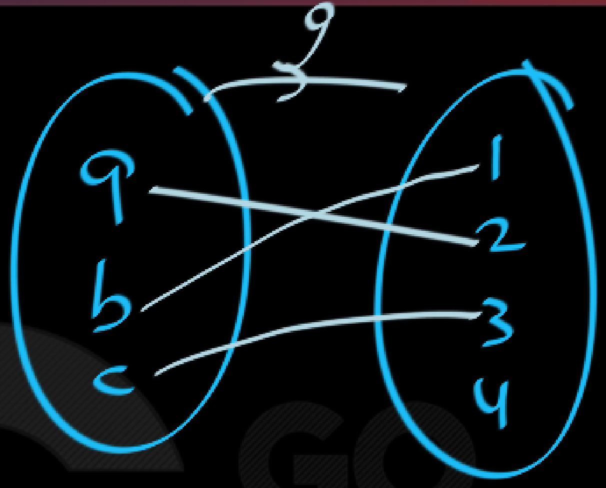
② Co-Domain(f) = Co-Domain(g)

i.e. $B = D$

③ Same mapping ; $\forall a \in$ Domain $(f(a) = g(a))$

Discrete Mathematics





$$\underbrace{g \neq f}$$

$$\underbrace{g(a) \neq f(a)}$$

for f, g :

Domain same
Co-Domain "
Range "
} still $f \neq g$

When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain. Note that if we change either the domain or the codomain

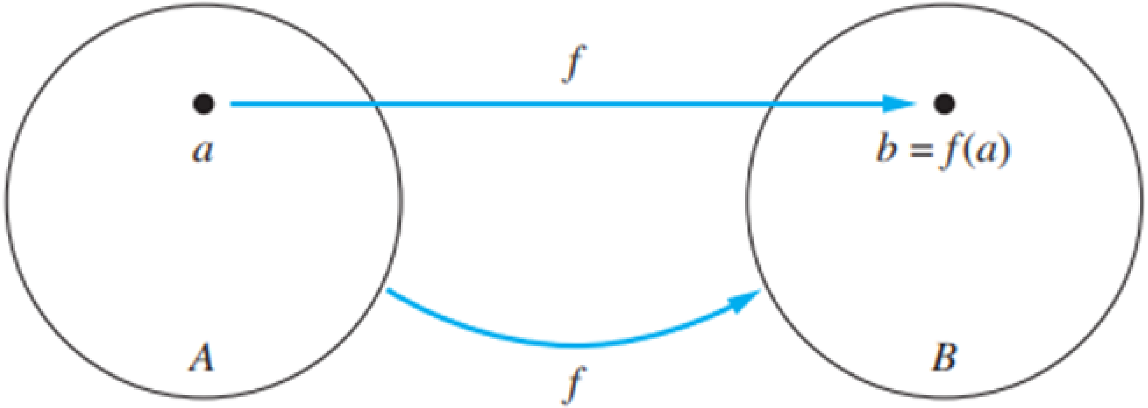
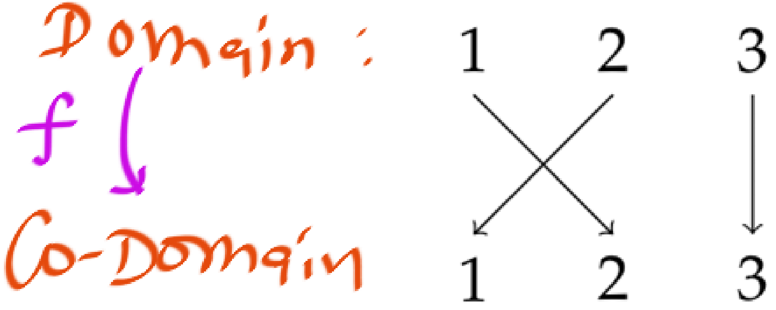


FIGURE 2 **The Function f Maps A to B.**

of a function, then we obtain a different function. If we change the mapping of elements, then we also obtain a different function.

Function Representations



$f(1)=2$

$f(2)=1$

$f(3)=3$

This shows that the function f sends 1 to 2, 2 to 1 and 3 to 3: just follow the arrows.

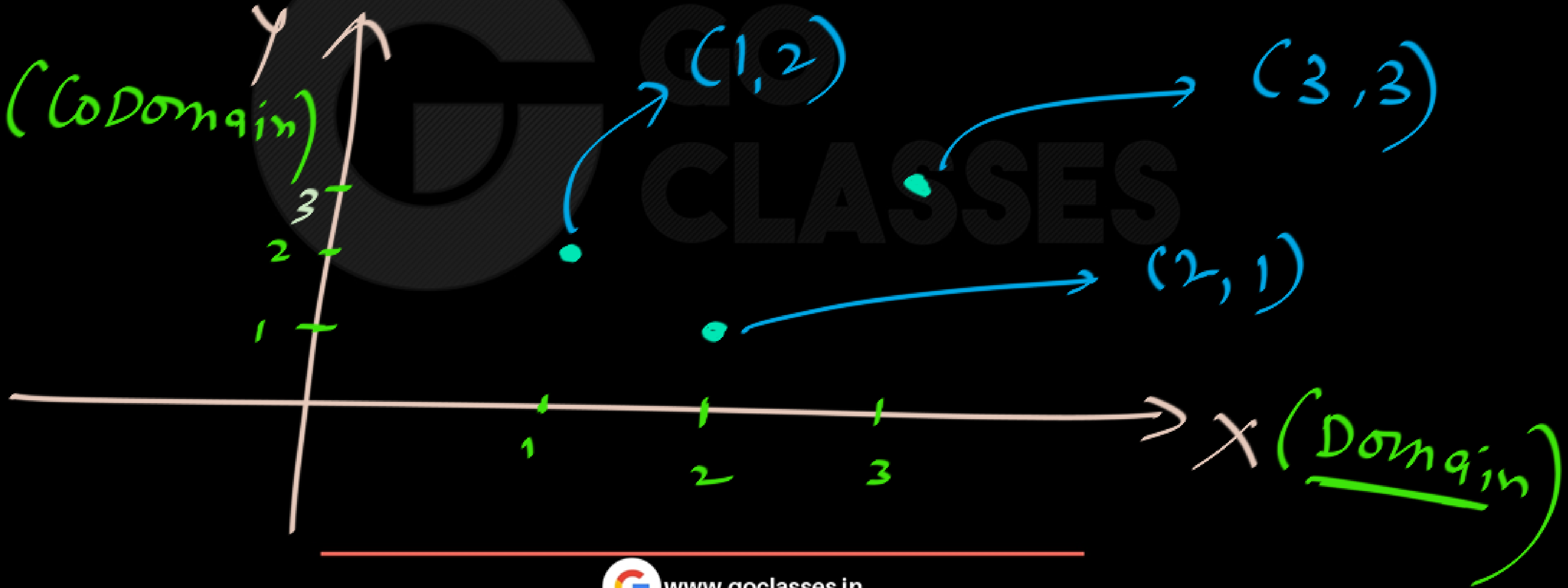
Set Representation : $f = \{(1,2),(2,1),(3,3)\}$

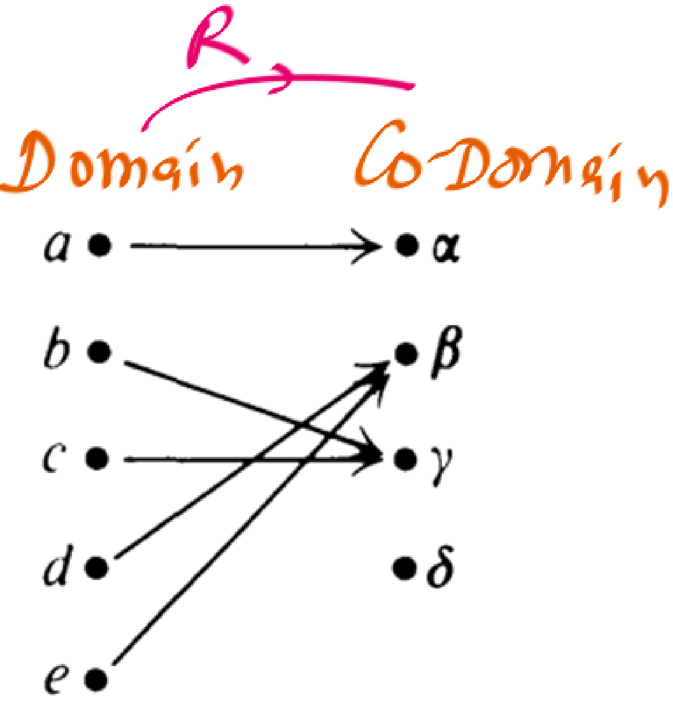
Table Rep :

f	1	2	3	: Domain
	2	1	3	: CoDomain

Graph Rep: $f : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$

$f(1) = 2$; $f(2) = 1$; $f(3) = 3$





(a)

Domain ↓

→ CoDomain

	α	β	γ	δ
a	✓			
b			✓	
c			✓	
d		✓		
e		✓		

(b) function R

	R
a	α
b	γ
c	γ
d	β
e	β

(c)

$f(0)=3$; $f(1)=3$; $f(2)=2$; $f(3)=4$

Domain →

→ x	0	1	2	3	4
→ $f(x)$	3	3	2	4	1

We simplify this further by writing this as a "matrix" with each input directly over its output:

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 & 1 \end{pmatrix}.$$

Another Function Representation:

Piecewise Function

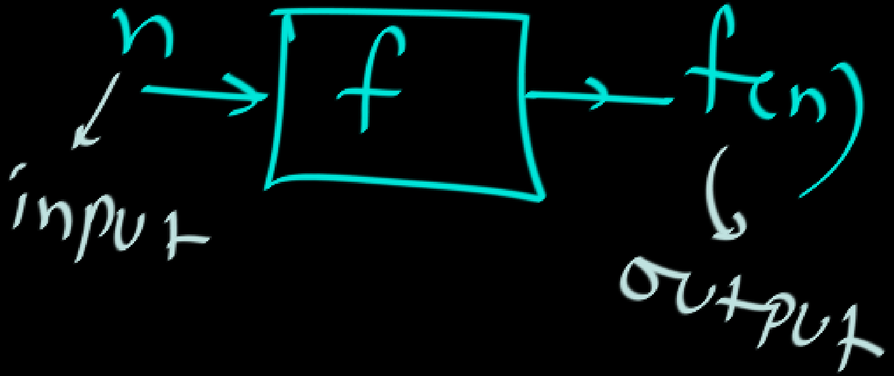
Piecewise Functions

- Functions may be specified **piecewise**, with different rules applying to different elements.
- Example:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{otherwise} \end{cases}$$

$f : Z \to Z$

$$\underbrace{f(n)}_{\text{output}} = \begin{cases} -\frac{n}{2} \ ; & n \text{ is even} \\ \frac{n+1}{2} \ ; & n \text{ odd} \end{cases} \Bigg\} \text{ input}$$



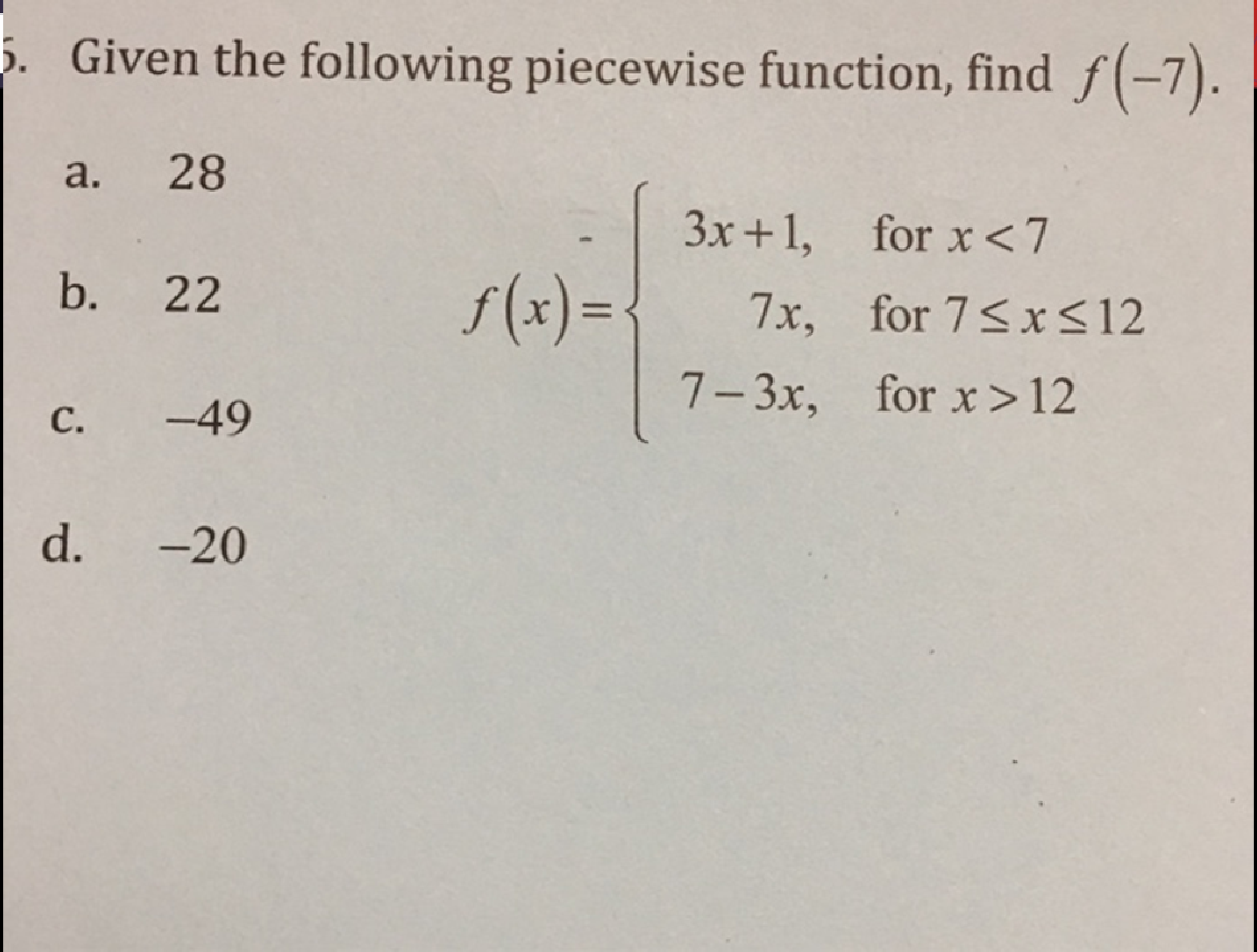
$\underset{\text{even}}{f(0)} = -\frac{0}{2} = 0$

$f(2) = -1$

$f(-4) = 2$

$f(\overset{\text{odd}}{3}) = \frac{3+1}{2} = 2$

$f(-3) = \frac{-3+1}{2} = -1$



5. Given the following piecewise function, find $f(-7)$.

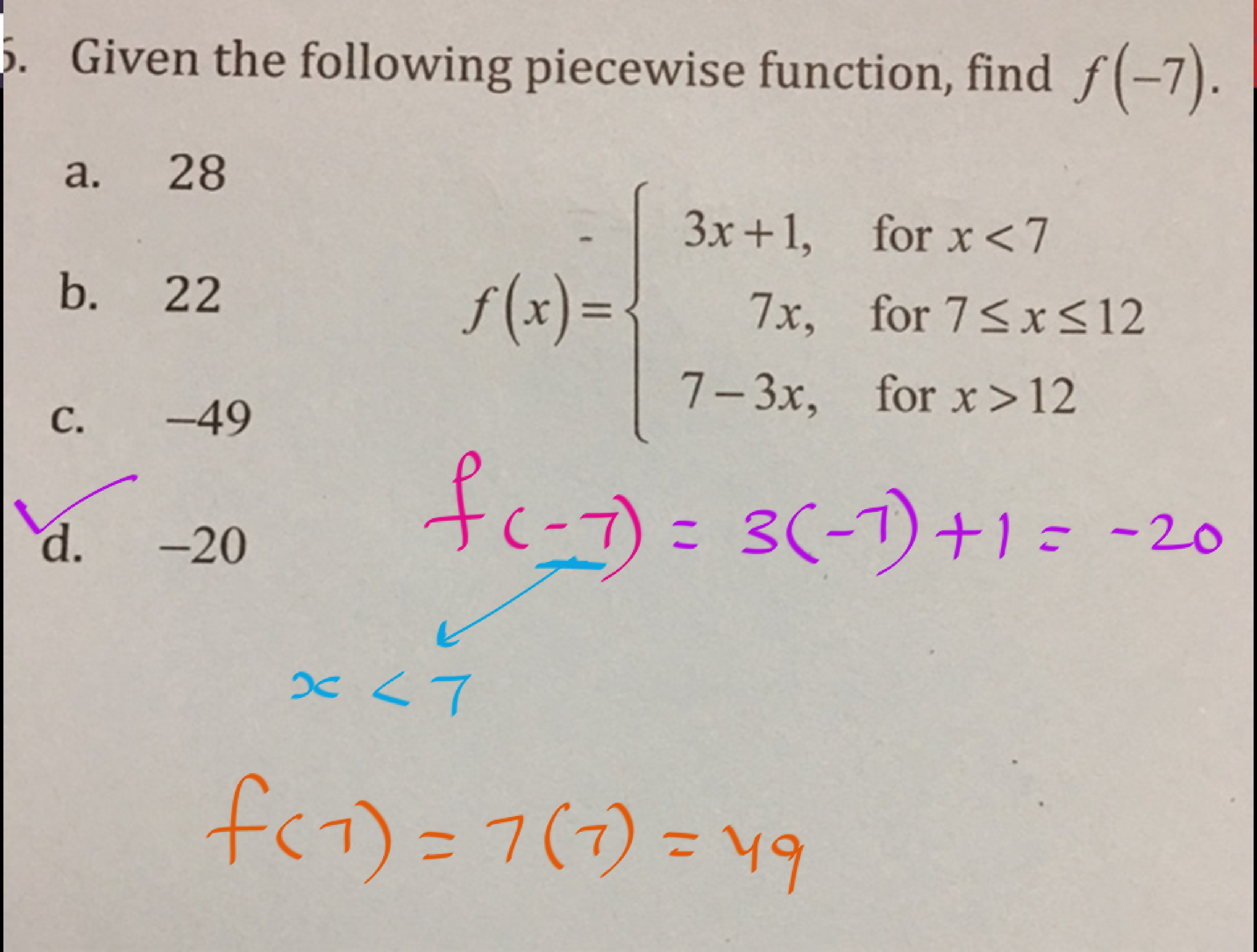
$$f(x)=\begin{cases} 3x+1, & \text{for } x<7 \\ 7x, & \text{for } 7\leq x\leq 12 \\ 7-3x, & \text{for } x>12 \end{cases}$$

a. 28

b. 22

c. –49

d. –20



5. Given the following piecewise function, find $f(-7)$.

a. 28

b. 22

c. –49

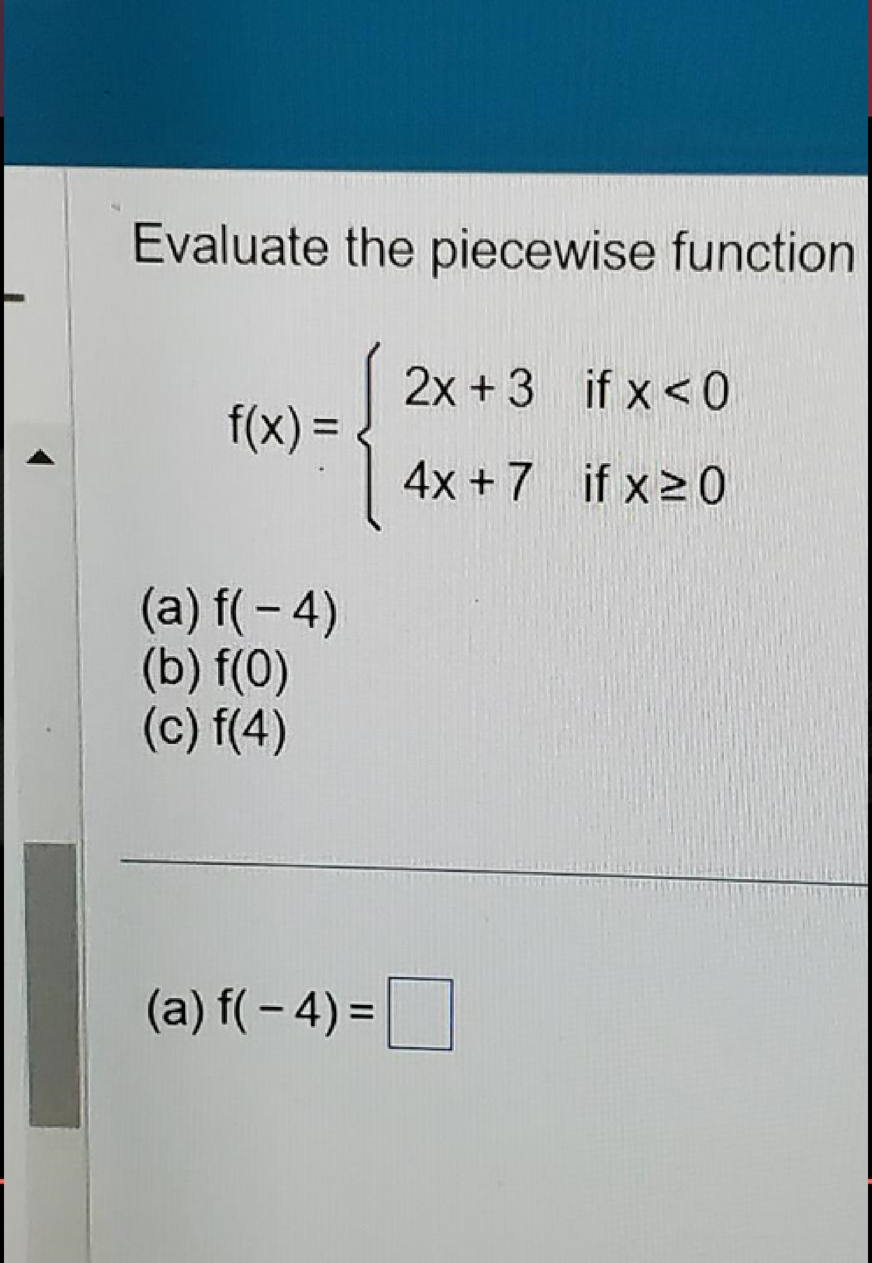
[✓ d. –20]

$$f(x)=\begin{cases} 3x+1, & \text{for } x<7 \\ 7x, & \text{for } 7\le x\le 12 \\ 7-3x, & \text{for } x>12 \end{cases}$$

$f(\underline{-7}) = 3(-7)+1 = -20$

$x < 7$

$f(7) = 7(7) = 49$

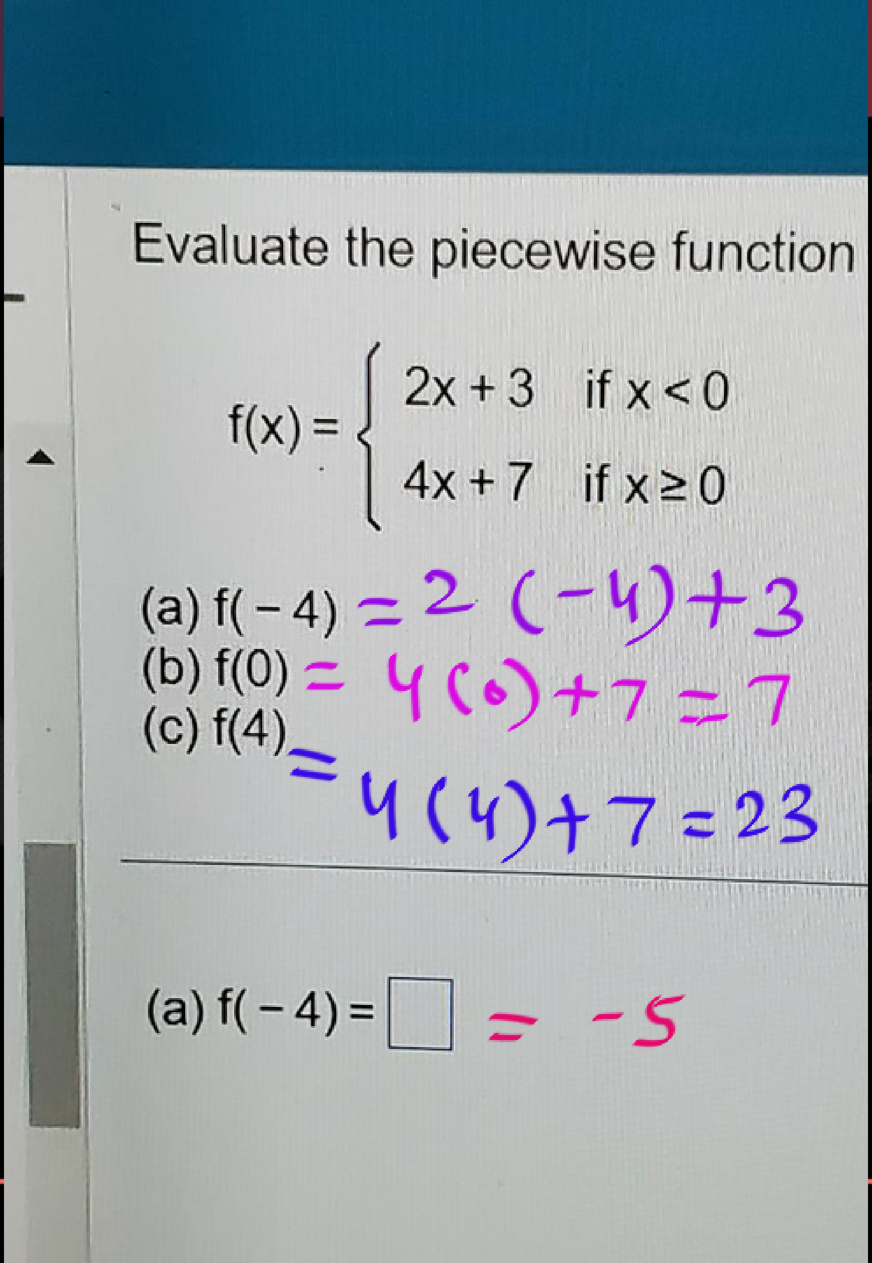


Evaluate the piecewise function

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 0 \\ 4x + 7 & \text{if } x \geq 0 \end{cases}$$

(a) $f(-4)$
(b) $f(0)$
(c) $f(4)$

(a) $f(-4) =$ ☐



Evaluate the piecewise function

$$f(x) = \begin{cases} 2x+3 & \text{if } x<0 \\ 4x+7 & \text{if } x \geq 0 \end{cases}$$

(a) $f(-4) = 2(-4)+3 = -5$

(b) $f(0) = 4(0)+7 = 7$

(c) $f(4) = 4(4)+7 = 23$

(a) $f(-4) = \square = -5$

Image of "a Subset of Domain"

- The function from $\{0, 1, 2, 3, 4\}$ to $\{a, b, c\}$ given by the following table:

Domain

Co-Domain

0	a
1	c
2	b
3	a
4	b

Range (f) : $\{a, b, c\}$ = Co-Domain

Image of 4 = b

Image(4) = b

Image$(\{3,4\}) = \{a, b\}$

Subset of Domain

Image$(\{2, 4\})$

$= \{b\}$

Image(Domain) = Range

$f: A \to B$; $S \subseteq A$

$\text{Image}(S) = \{ f(x) \mid x \in S \}$

$\text{Image}(S) = \{ y \mid y \in B ;\ \exists x \in S\ f(x) = y \}$

Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

Types of Functions:

Injective, Surjective & Bijective Function

Injective Function

One-to-One function / Injective / Injection

Injective Function:

Different elements of domain have different images.

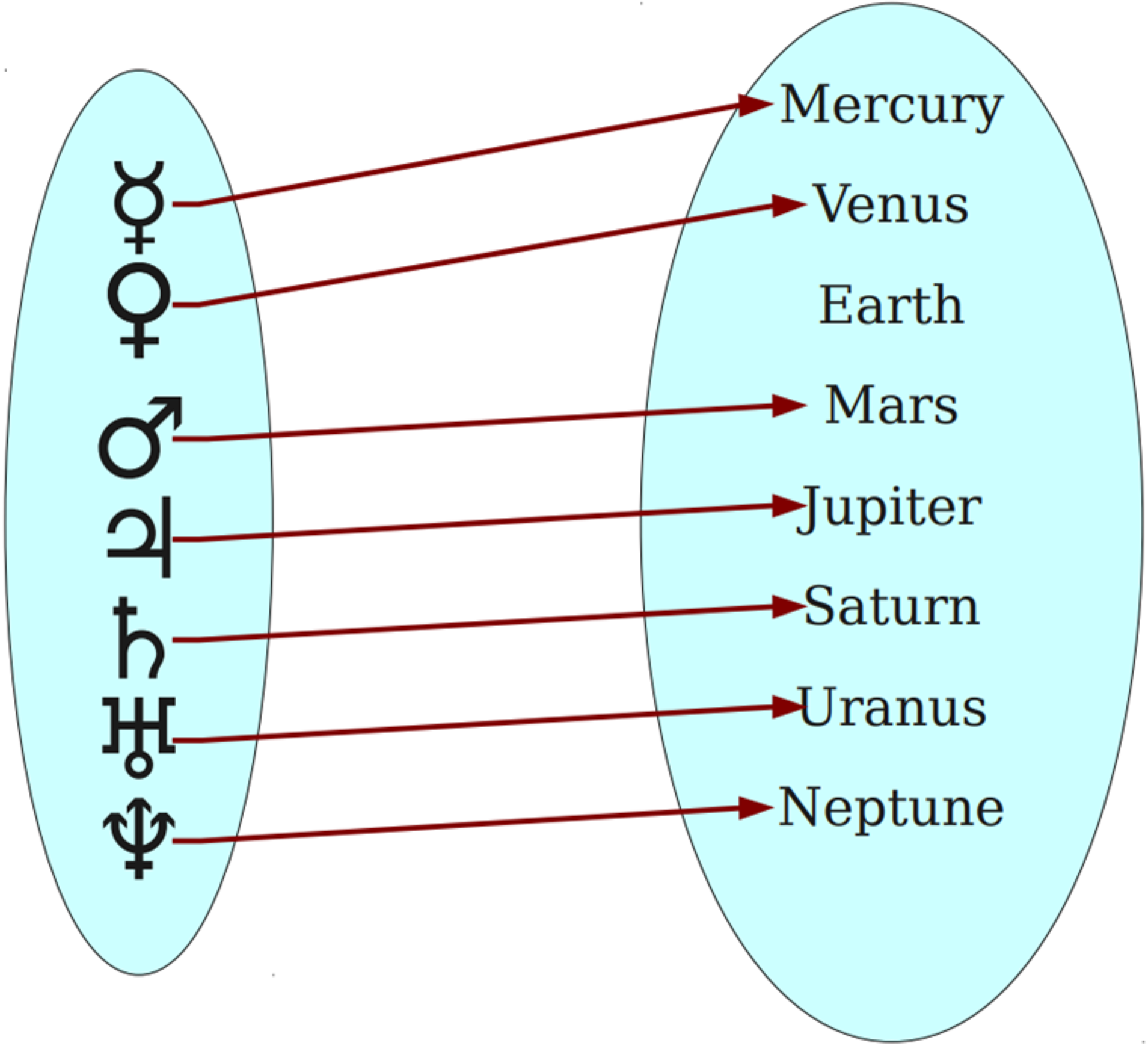
Injective Function:

Different elements of domain have different images.

$f : A \longrightarrow B$; f is Injection (one-to-one)

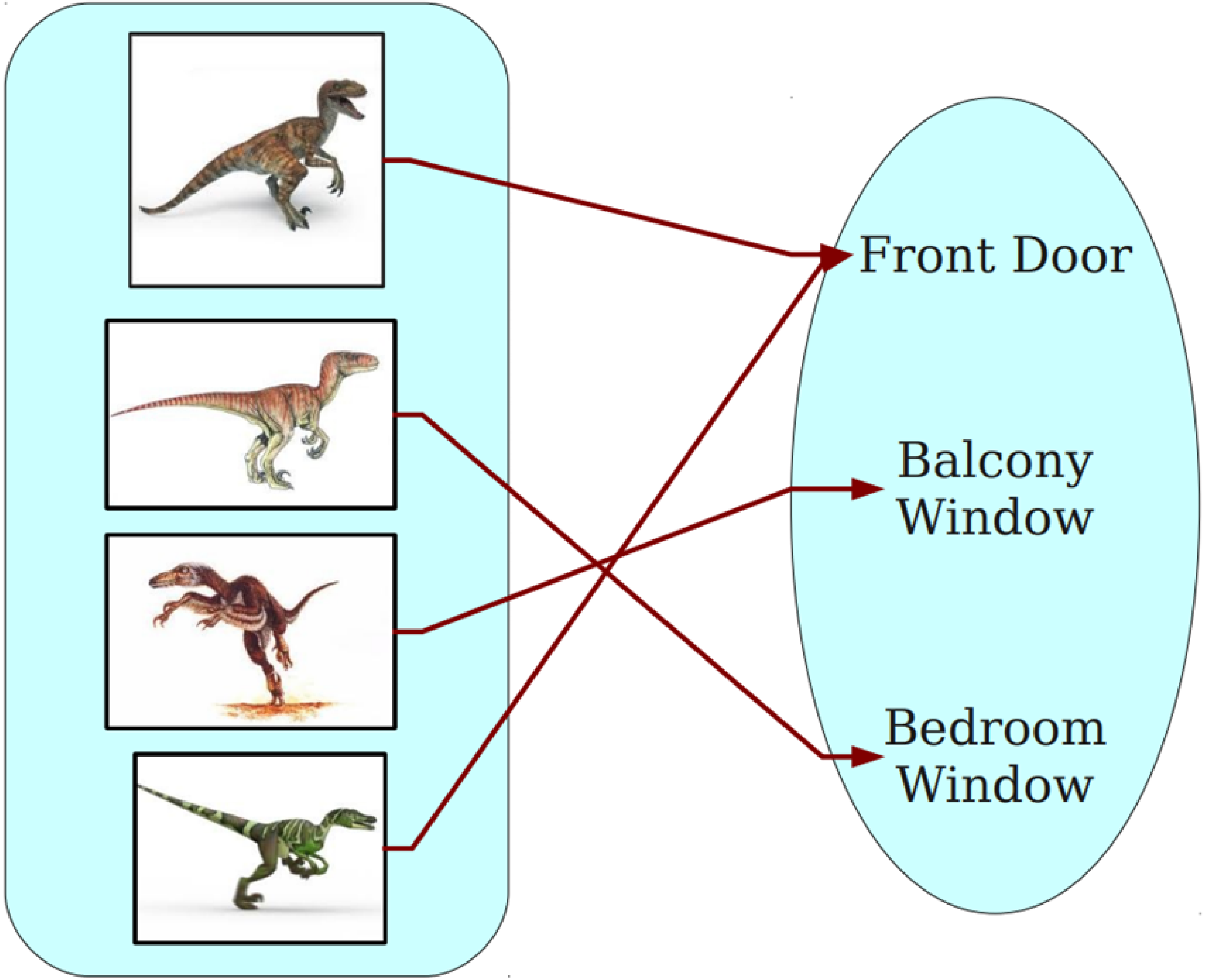
iff $\forall_{\underline{x, y \in A}} \left((x \neq y) \longrightarrow (f(a) \neq f(b)) \right)$ ✓

iff $\forall_{x, y \in A} \left((f(a) = f(b)) \Longrightarrow (a = b) \right)$ ✓



Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective function ✓

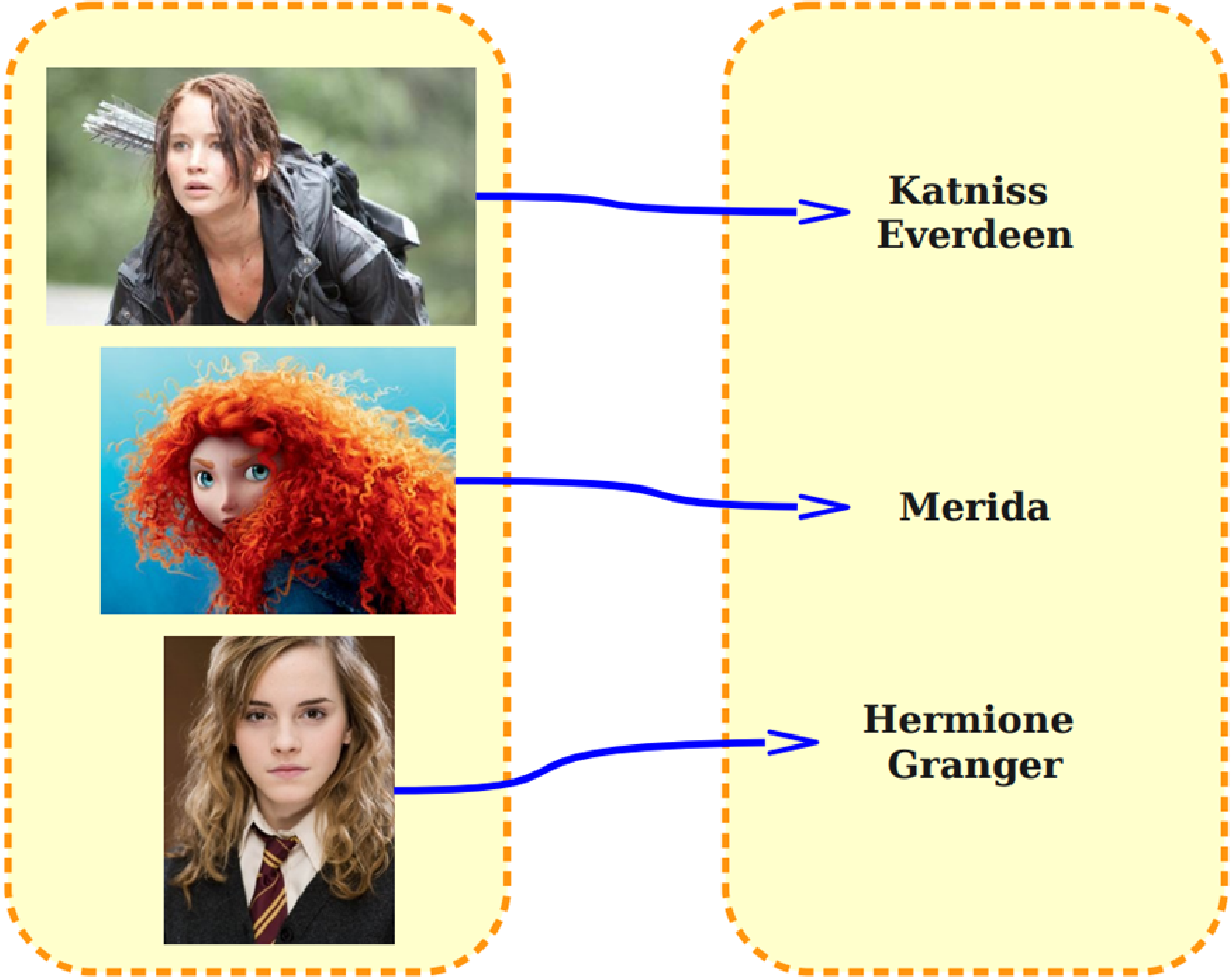


Front Door

Balcony Window

Bedroom Window

NOT one-to-one function



Katniss Everdeen

Merida

Hermione Granger

Injective function ✓

$f(x) = x^2$

$1 \rightarrow 1$
$2 \rightarrow 4$
$3 \rightarrow 9$ . . . - -

$f : N \rightarrow N$: f : Injective ✓

$f : Z \rightarrow Z$: f : NOT Injection

$f(2) = f(-2) = 4$

3.5.5.2 Injections

If $f : A \to B$ maps distinct elements of A to distinct elements of B (i.e., if $x \neq y$ implies $f(x) \neq f(y)$), it is called **one-to-one**, **injective**, or an **injection**. By contraposition, an equivalent definition is that $f(x) = f(y)$ implies $x = y$ for all x and y in the domain. For example, the function $f(x) = x^2$ from $\mathbb{N}$ to $\mathbb{N}$ is injective. The function $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$ is *not* injective (for example, $f(-1) = f(1) = 1$). The function $f(x) = x + 1$ from $\mathbb{N}$ to $\mathbb{N}$ is injective.

One-to-One Functions

Let F be a function from a set X to a set Y. F is **one-to-one** if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2$$

or

$$\forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

The function,

$$f = (1, b), (2, a), (3, c)$$

from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is one-to-one. {*from Johnsonbaugh, p. 119*}

$g : R \rightarrow R$

$g(x) = x^4 - x$

Injective function ? **NO**

$$\boxed{f(0) = f(1) = 0}$$

Counter Example

$f(2) = 14$

$f(-2) = 18$

Consider the function $f : R \to R$ where $f(x) = 3x + 7$ for all $x \in R$. Then for all $x_1, x_2 \in R$, we find that

$$f(x_1) = f(x_2) \Rightarrow 3x_2 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2,$$

so the given function is one-to-one.

On the other hand, suppose that $g : R \to R$ is the function defined by $g(x) = x^4 - x$ for each real number x. Then

$$g(0) = (0)^4 - 0 = 0 \text{ and } (1)^4 - 1 = 1 - 1 = 0.$$

Consequently, g is **not** one-to-one since $g(0) = g(1)$ but $0 \neq 1$. { from Grimaldi, p 255}

Injective Function Definition from Co-domain perspective:

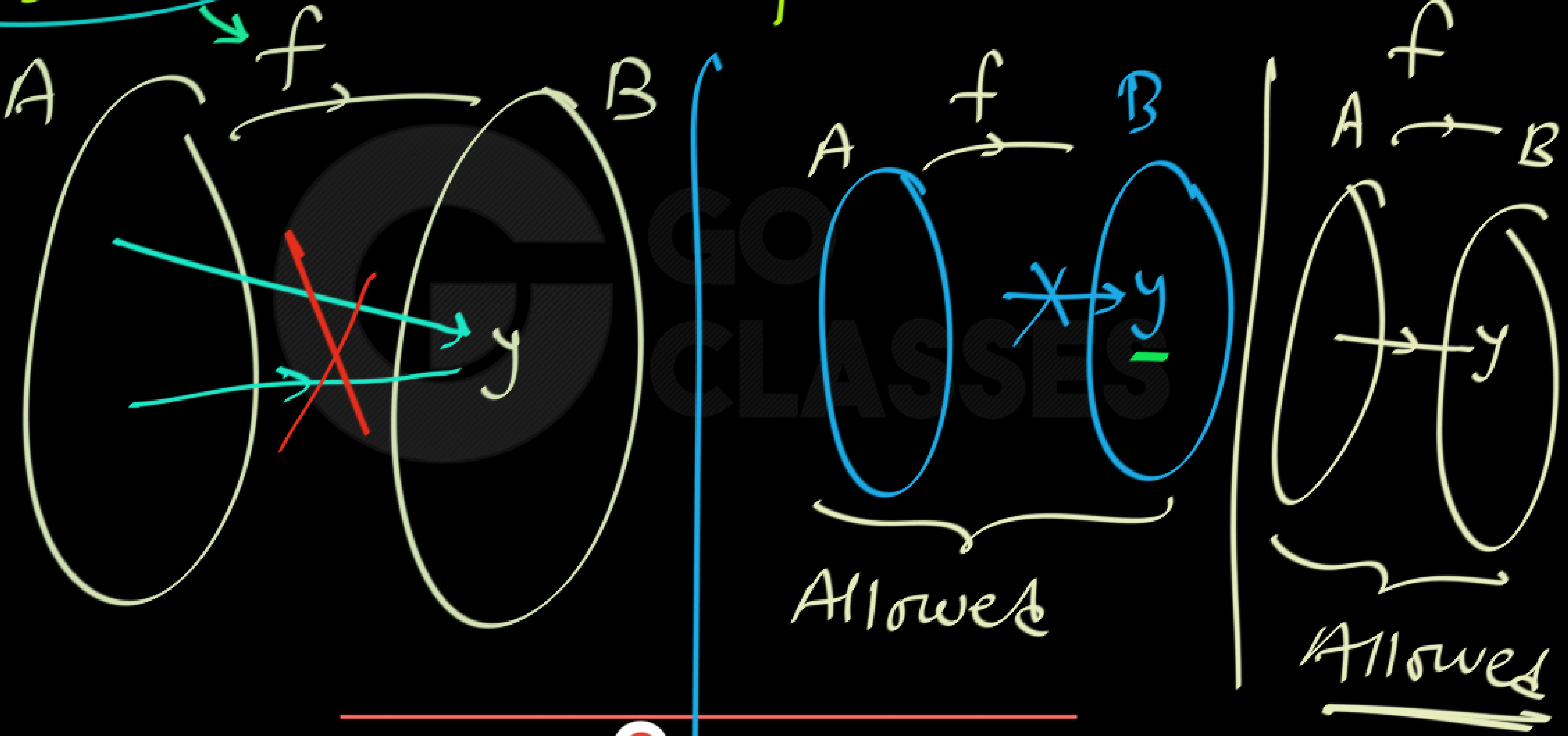
Which of the following is another definition of Injective Function?

A. Each element of the codomain has at least one pre-image in the domain.

B. Each element of the codomain has at most one pre-image in the domain.

C. Each element of the codomain has exactly one pre-image in the domain.

Injective function $f : A \to B$



Injective Function Definition from Co-domain perspective:

Which of the following is another definition of Injective Function?

A. Each element of the codomain has at least one pre-image in the domain.

[✓ B.] Each element of the codomain has at most one pre-image in the domain. ✓

C. Each element of the codomain has exactly one pre-image in the domain.

Injective Functions

- A function $f : A \to B$ is called **injective** (or **one-to-one**) if each element of the codomain has at most one element of the domain associated with it.
  - A function with this property is called an **injection**.
- Formally:

$$\textbf{If } f(x_0) = f(x_1)\textbf{, then } x_0 = x_1$$

Surjective Function

Onto function / Surjective / Surjection

Surjective Function:

Every element of Co-domain is covered.

Range = Co-domain

Surjective Function:

Every element of codomain has at least one pre-image.

$f : A \rightarrow B$; f is onto iff

$\forall y \in B$ (there is at least one element x in Domain which maps to y)

Co-Domain

Surjective Function:

Every element of codomain has at least one pre-image.

$f : A \rightarrow B$; f is onto iff

$$\forall_{y \in B} \; \exists_{x \in A} \left(f(x) = y \right)$$

Co-Domain

Surjective Functions

- A function $f : A \to B$ is called **surjective** (or **onto**) if each element of the codomain has at least one element of the domain associated with it.
  - A function with this property is called a **surjection**.
- Formally:

For any $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

- An intuition: surjective functions cover every element of B with at least one element of A.

$$\left.\begin{array}{l} f : Z \to Z \\ f(x) = x^2 \end{array}\right\} \longrightarrow \underline{\text{not onto.}}$$

No pre-image for 3.

$$\underbrace{\text{Range}}_{\{0, 1^2, 2^2, 3^2, 4^2, \ldots\}} \neq \underbrace{\text{Co-Domain}}_{Z}$$

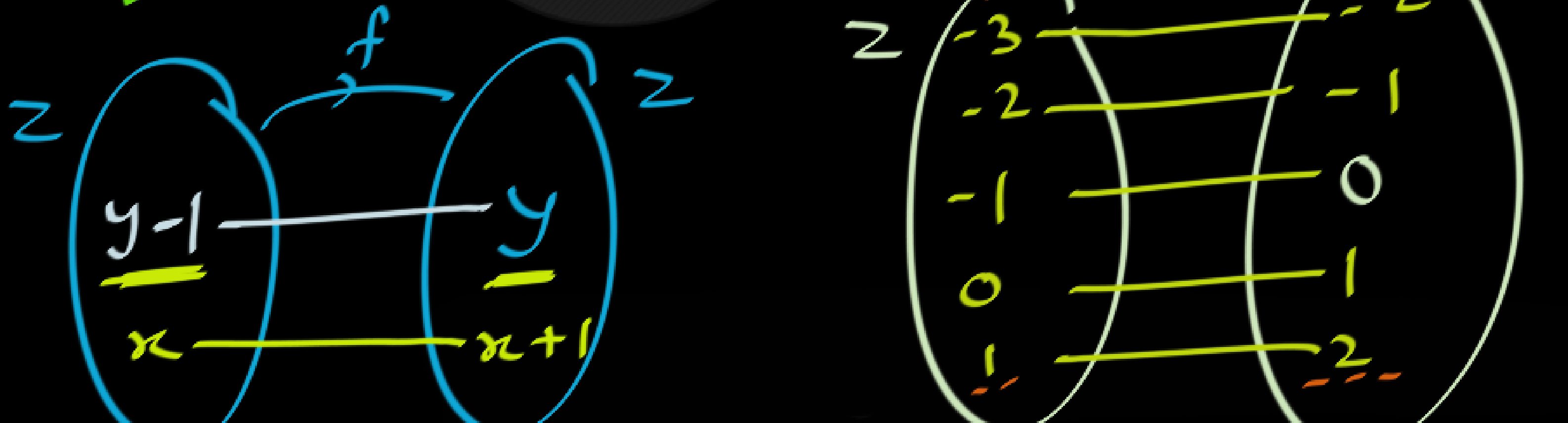
$$\{0, 1^2, 2^2, 3^2, 4^2, \ldots\} \neq Z$$

$f(x) = x+1$

$N = \{1, 2, 3, --\}$

$f : N \to N$ ⟶ not onto (No preimage of 1)

$\underbrace{f : Z \to Z}$ ⟶ onto ✓



Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance. ◀

Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

3.5.5.1 Surjections

A function $f : A \to B$ that covers every element of B is called **onto**, **surjective**, or a **surjection**. This means that for any y in B, there exists some x in A such that $y = f(x)$. An equivalent way to show that a function is surjective is to show that its **range** $\{f(x) \mid x \in A\}$ is equal to its codomain.

For example, the function $f(x) = x^2$ from $\mathbb{N}$ to $\mathbb{N}$ is not surjective, because its range includes only perfect squares. The function $f(x) = x + 1$ from $\mathbb{N}$ to $\mathbb{N}$ is not surjective because its range doesn't include 0. However, the function $f(x) = x + 1$ from $\mathbb{Z}$ to $\mathbb{Z}$ *is* surjective, because for every y in $\mathbb{Z}$ there is some x in $\mathbb{Z}$ such that $y = x + 1$.

Bijective Function

Bijection / "One-to-One Correspondence"

Bijective Function:

Bijective = “Injective & Surjective”

Injective Function:

Every element of codomain has At Most one pre-image.

Surjective Function:

Every element of codomain has At Least one pre-image.

Bijective Function:

Every element of codomain has ?????? pre-image.

Injective Function:

Every element of codomain has At Most one pre-image.

Surjective Function:

Every element of codomain has At Least one pre-image.

Bijective Function:

Every element of codomain has Exactly one pre-image.

Injections and Surjections

- An injective function associates **at most** one element of the domain with each element of the codomain.
- A surjective function associates **at least** one element of the domain with each element of the codomain.
- What about functions that associate **exactly one** element of the domain with each element of the codomain?

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called **bijective**.
  - Such a function is a **bijection**.
- Formally, a bijection is a function that is both **injective** and **surjective**.
- A bijection is a one-to-one correspondence between two sets.

Injections and Surjections

- A function $f : A \to B$ is an **injection** iff

  **for any $a_0, a_1 \in A$:
  if $f(a_0) = f(a_1)$, then $a_0 = a_1$.**

- *At most* one element of the domain maps to each element of the codomain.
- A function $f : A \to B$ is a **surjection** iff

  for any $b \in B$, there exists an $a \in A$ where $f(a) = b$.

- *At least* one element of the domain maps to each element of the codomain.

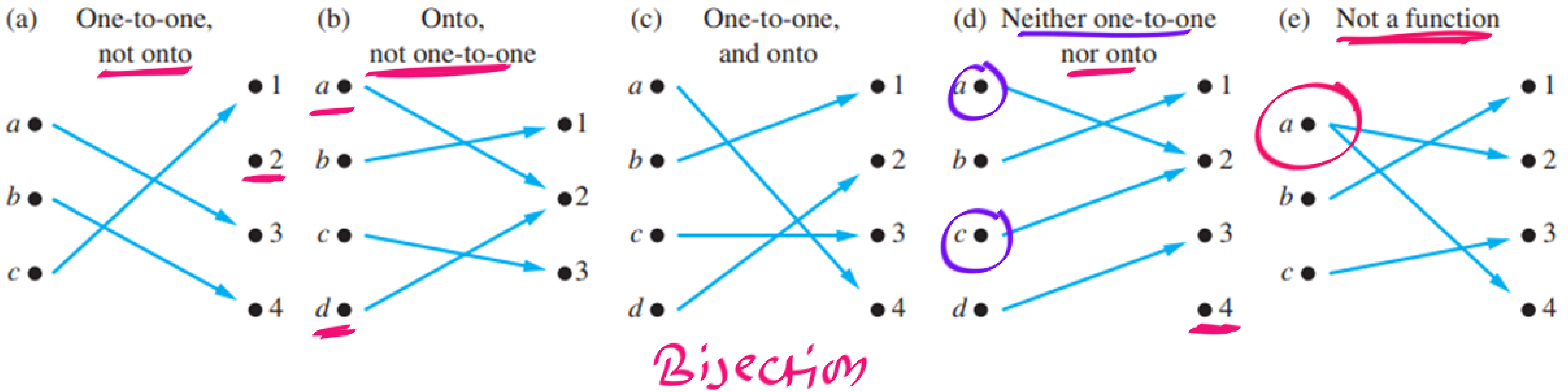


FIGURE 5 Examples of Different Types of Correspondences.

Functions & Cardinalities

Injective Functions & Cardinalities:

Injective Functions & Cardinalities:

Q:

If |Domain| > |Co-Domain|

Can we create some Injective function??

Injective Functions & Cardinalities:

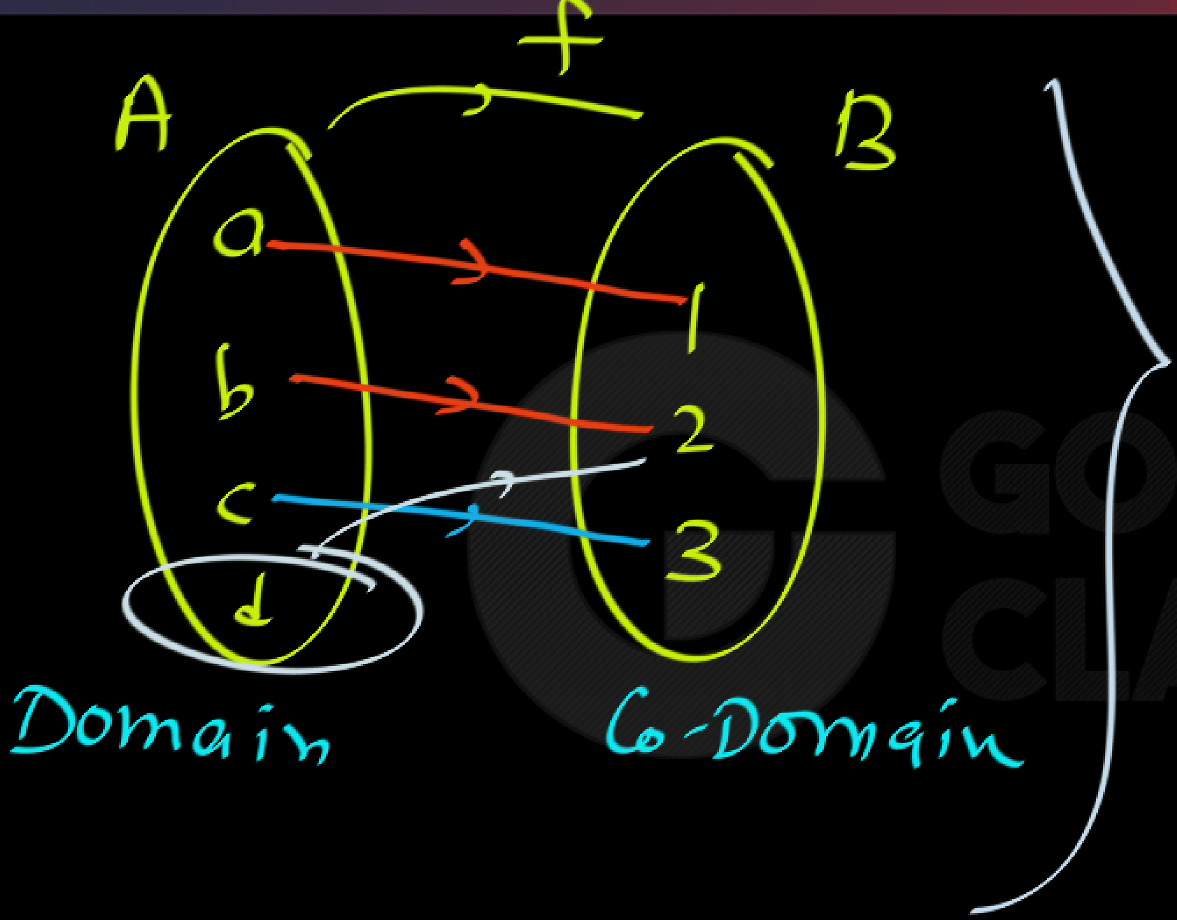
Q:

If |Domain| > |Co-Domain|

Can we create "some" Injective function??

No

No Injective function Possible.



No Injective function from A to B possible.

Injective Functions & Cardinalities:

Q:

If |Domain| <= |Co-Domain|

Can we create some Injective function??

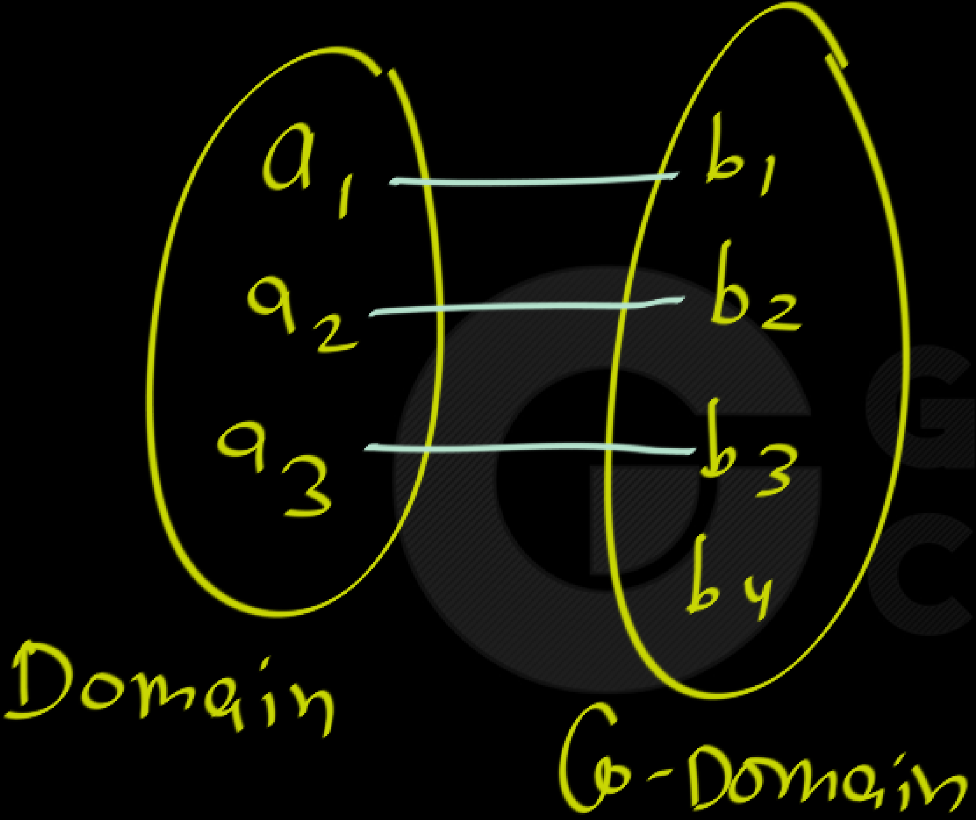
Injective Functions & Cardinalities:

Q:

If |Domain| <= |Co-Domain|

Can we create some Injective function??

YES.



Some Injective fun. Possible.

Injective Functions & Cardinalities:

Theorem:

✓

Injective Function EXISTS

iff

|Domain| <= |Co-Domain|

Let S, T be ANY two sets (finite or infinite)

$$|S| <= |T|$$

iff

There Exists a Injection from S to T.

Q:

f is a function from Set X to Set Y, which is NOT Injective. Then ??

A. |X| > |Y|
B. |X| <= |Y|
C. Can’t say anything

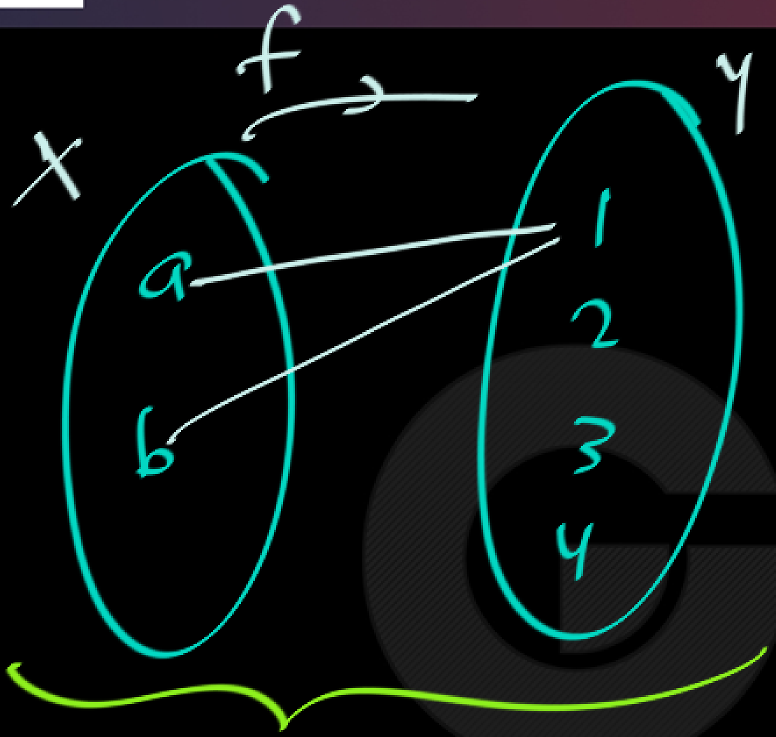
Q:

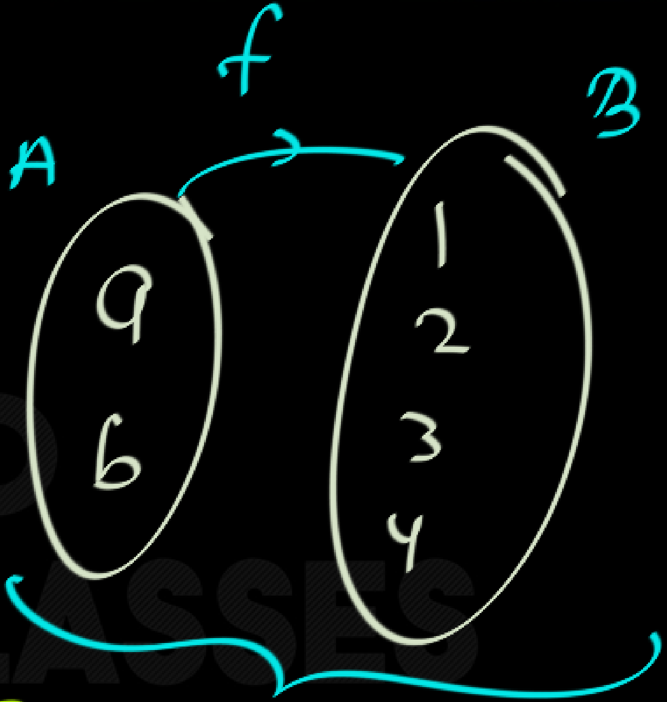
f is a function from Set X to Set Y, which is NOT Injective. Then ??

✗ A. |X| > |Y|

✗ B. |X| <= |Y|

C. Can’t say anything ✓





Some injective fun. Exists.

Injective Functions & Cardinalities:

Theorem:

Injective Function EXISTS
iff
|Domain| <= |Co-Domain|

Q:

f is a function from Set X to Set Y, which is Injective. Then ??

A. |X| > |Y|
B. |X| <= |Y|
C. Can’t say anything

Q:

f is a function from Set X to Set Y, which is Injective. Then ??

✗ A. |X| > |Y|

[✓ B. |X| <= |Y|]

C. Can’t say anything

Q:

There is NO Injective function from Set X to Set Y. Then ??

A. |X| > |Y|
B. |X| <= |Y|
C. Can’t say anything

Q:

There is NO Injective function from Set X to Set Y. Then ??

A. |X| > |Y| ✓

B. |X| <= |Y|

C. Can’t say anything

Surjective Functions & Cardinalities:

Surjective Functions & Cardinalities:

Q:

If |Domain| < |Co-Domain|

Can we create some Surjective function??

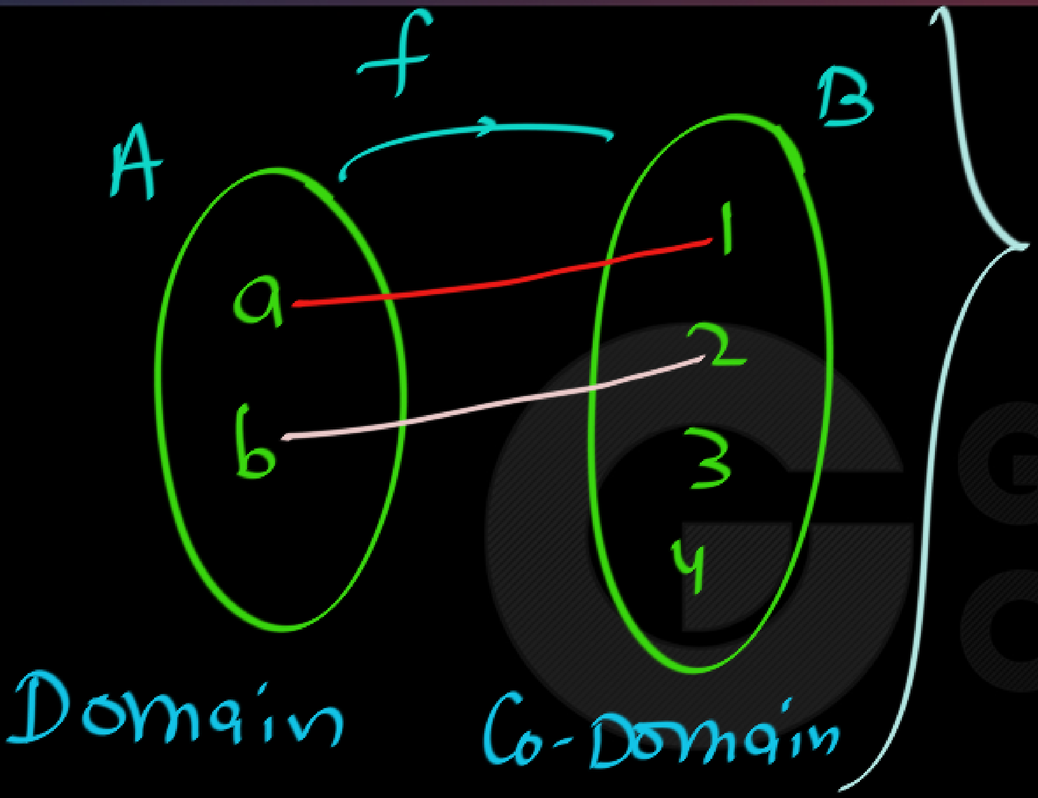
Surjective Functions & Cardinalities:

Q:

If |Domain| < |Co-Domain|

Can we create some Surjective function??

No.



No onto function Possible.

Surjective Functions & Cardinalities:

Q:

If |Domain| >= |Co-Domain|

Can we create some Surjective function??

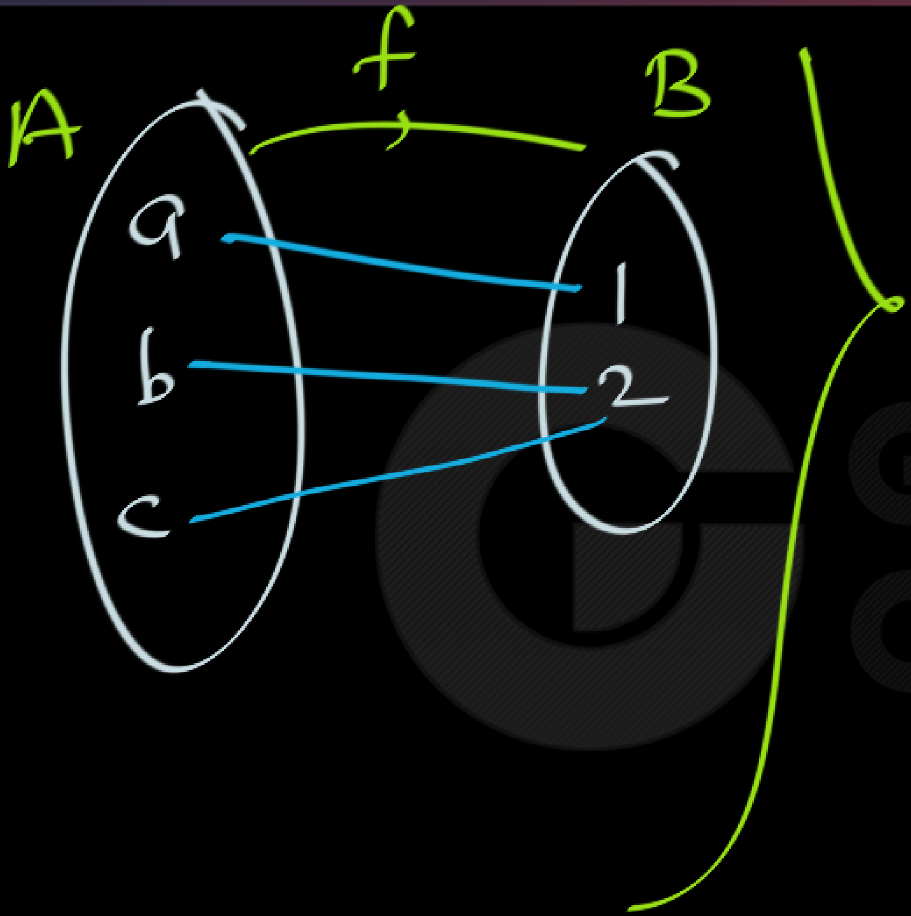
Surjective Functions & Cardinalities:

Q:

If |Domain| >= |Co-Domain|

Can we create some Surjective function??

YES.



Some onto function Possible.

Surjective Functions & Cardinalities:

Theorem:

Surjective Function EXISTS

iff

|Domain| >= |Co-Domain|

Let S, T be ANY two sets (finite or infinite)

|S| >= |T|

iff

There Exists a Surjection from S to T.

Q:

f is a function from Set X to Set Y, which is NOT Surjective. Then ??

A. $|X| < |Y|$
B. $|X| >= |Y|$
C. Can’t say anything

Q:

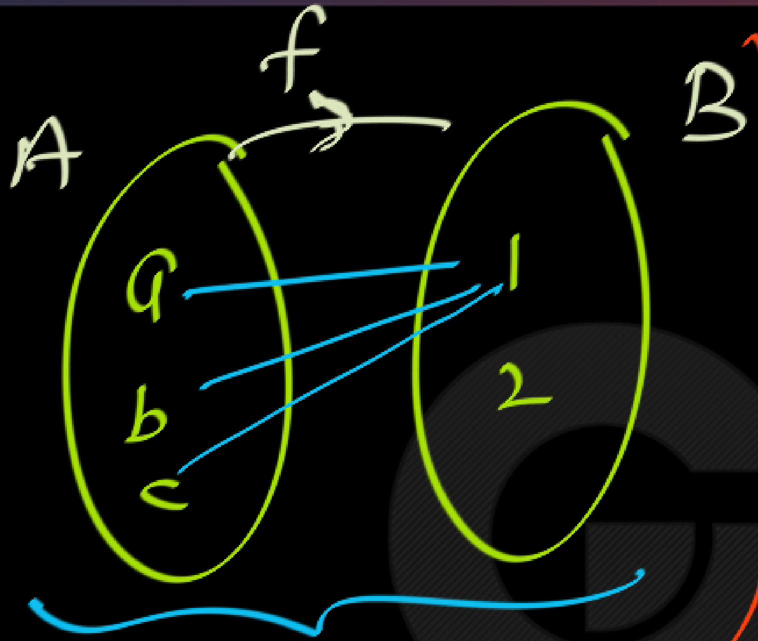
f is a function from Set X to Set Y, which is NOT Surjective. Then ??

✗ A. $|X| < |Y|$

✗ B. $|X| >= |Y|$

C. Can’t say anything ✓

Discrete Mathematics



not onto

Some onto function Possible because

$|\text{Domain}| \geq |\text{Co-Domain}|$

Let S, T be ANY two sets (finite or infinite)

$$|S| >= |T|$$

iff

There Exists a Surjection from S to T.

Q:

f is a function from Set X to Set Y, which is Surjective. Then ??

A. |X| < |Y|
B. |X| >= |Y|
C. Can’t say anything

Q:

f is a function from Set X to Set Y, which is Surjective. Then ??

A. $|X| < |Y|$

[✓ B. $|X| >= |Y|$]

C. Can’t say anything

Q:

There is NO Surjective function from Set X to Set Y. Then ??

A. |X| < |Y|
B. |X| >= |Y|
C. Can’t say anything

Q:

There is NO Surjective function from Set X to Set Y. Then ??

✓ A. |X| < |Y| ✓

B. |X| >= |Y|

C. Can’t say anything

Bijective Functions & Cardinalities:

Bijective Functions & Cardinalities:

Q:

If |Domain| > |Co-Domain|

Can we create some Bijective function??

Bijective Functions & Cardinalities:

Q:

If |Domain| > |Co-Domain|

→ Injective & Surjective

Can we create some Bijective function??

No. Injective function NOT possible.

Bijective Functions & Cardinalities:

Q:

If |Domain| < |Co-Domain|

Can we create some Bijective function??

Bijective Functions & Cardinalities:

Q:

If |Domain| < |Co-Domain|

1-1 & onto

Can we create some Bijective function??

No. because No surjective function possible.

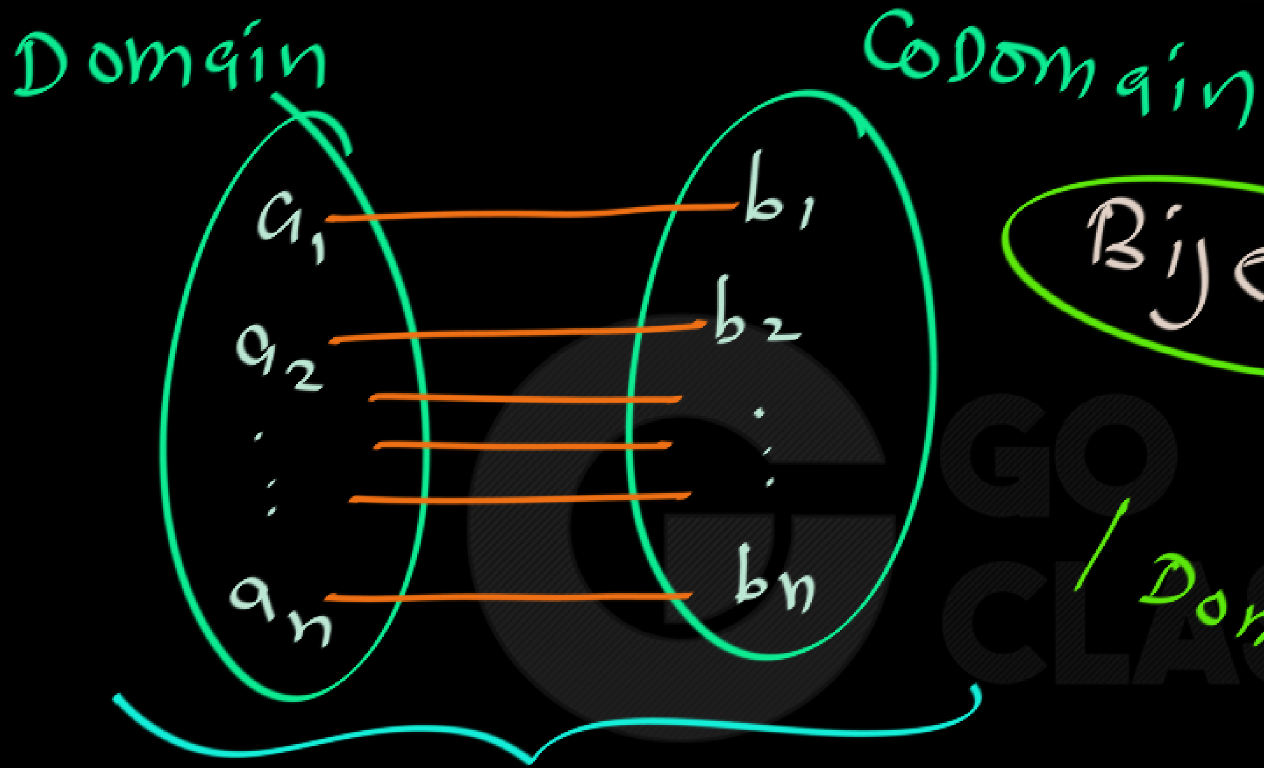
Bijective Functions & Cardinalities:

Q:

If $\underline{|\text{Domain}|} = \underline{|\text{Co-Domain}|}$

Can we create some Bijective function??

YES.



Bijection

(1-1 Correspondance)

Bijection Exists iff $|\text{Domain}| = |\text{Co-Domain}|$

Bijective Functions & Cardinalities:

Bijection = Injection & Surjection ✓

$|\text{Domain}| = |\text{Co-Domain}|$

$|\text{Domain}| \le |\text{Co-Domain}|$ & $|\text{Domain}| \ge |\text{Co-Domain}|$

Bijective Functions & Cardinalities:

Theorem:

Bijective Function EXISTS

iff

|Domain| = |Co-Domain|

Let S, T be ANY two sets (finite or infinite)

$$|S| = |T|$$

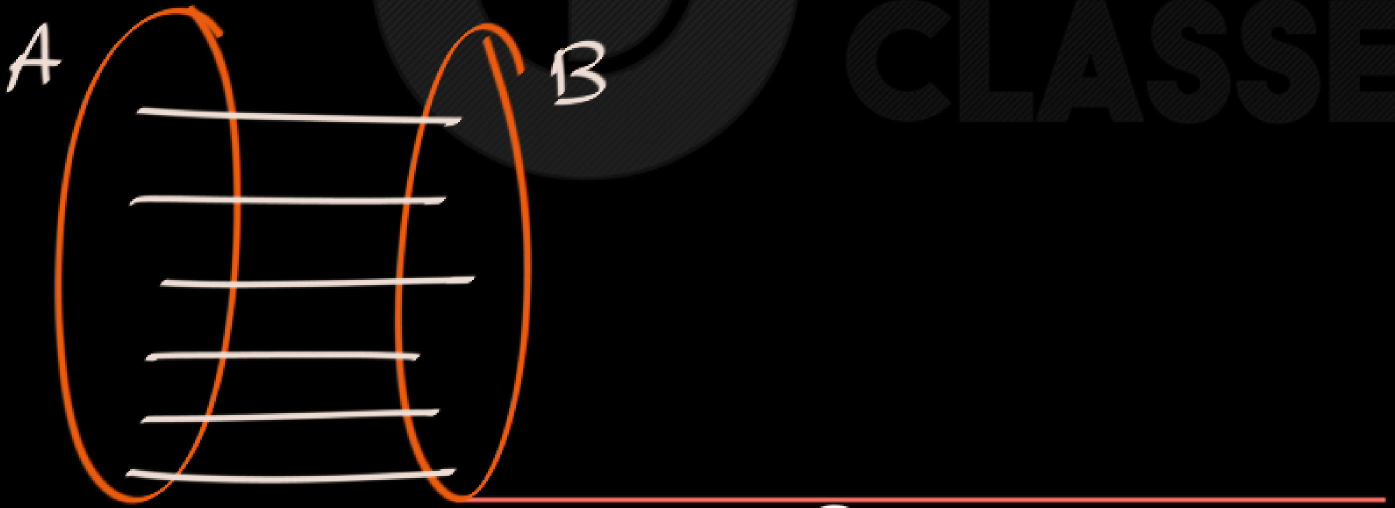
iff

There Exists a Bijection from S to T.

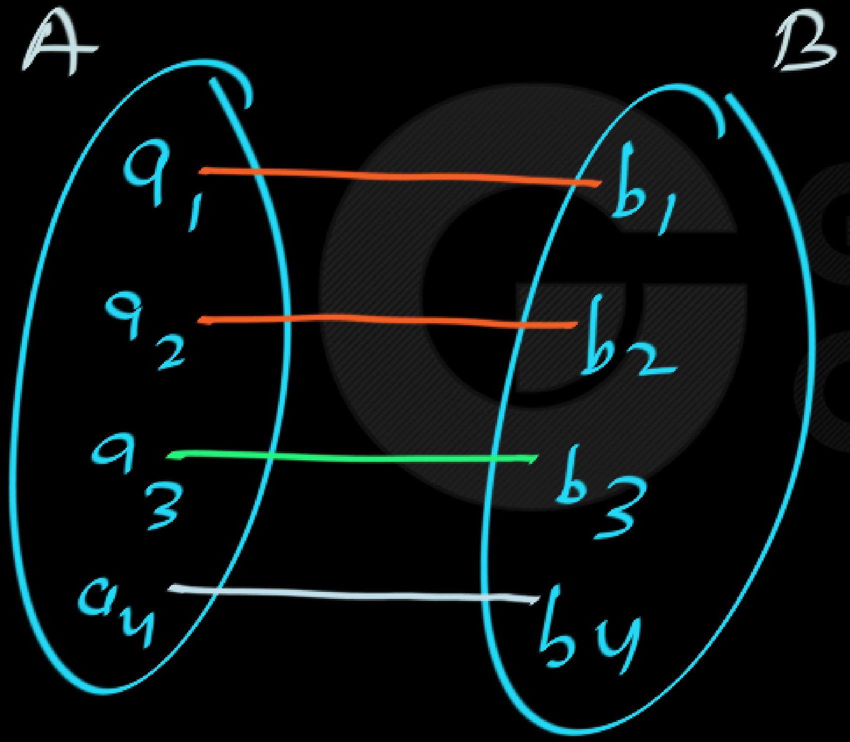
Bijection exists iff |Domain| = |Co-Domain|

Why Bijection is called

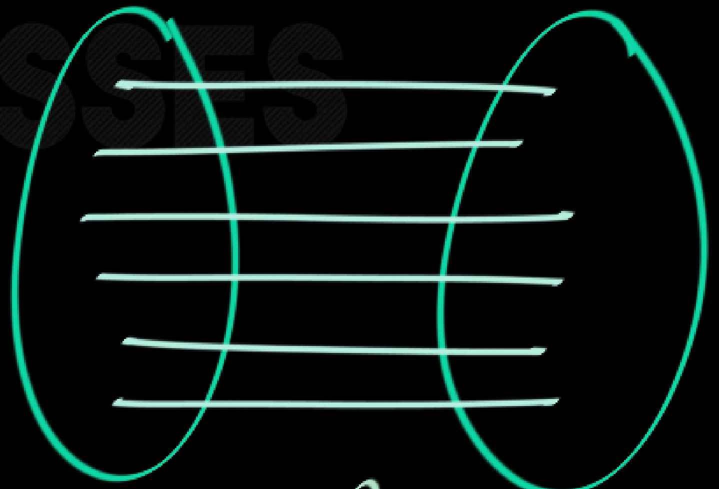
One-to-one Correspondance ?



Bijection exists iff |Domain| = |Co-Domain|



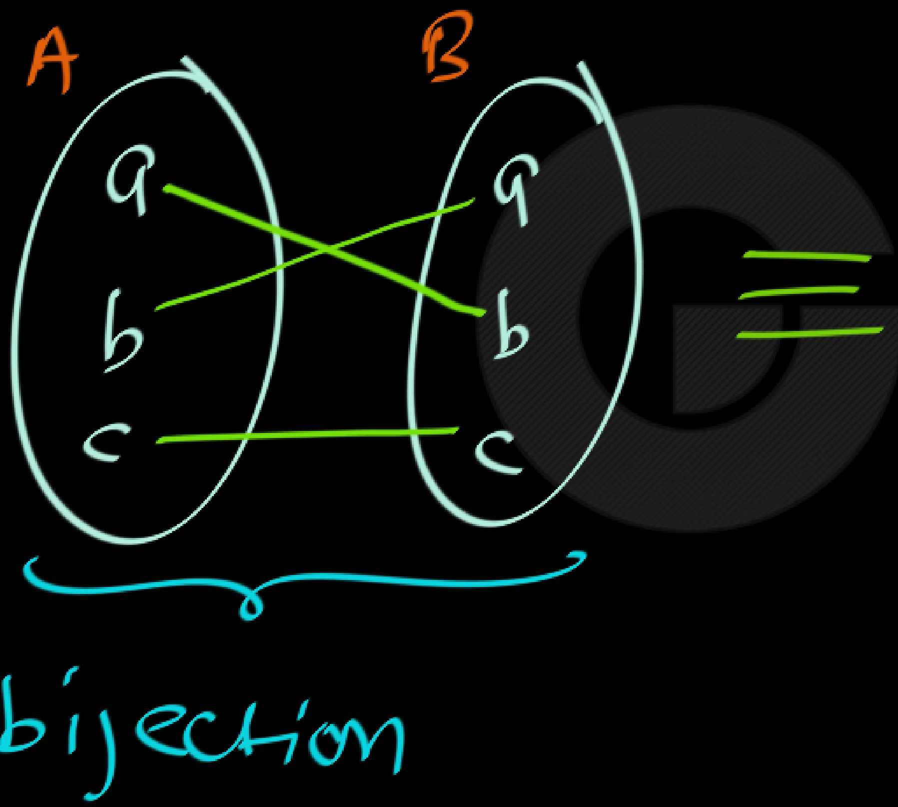
Bijection

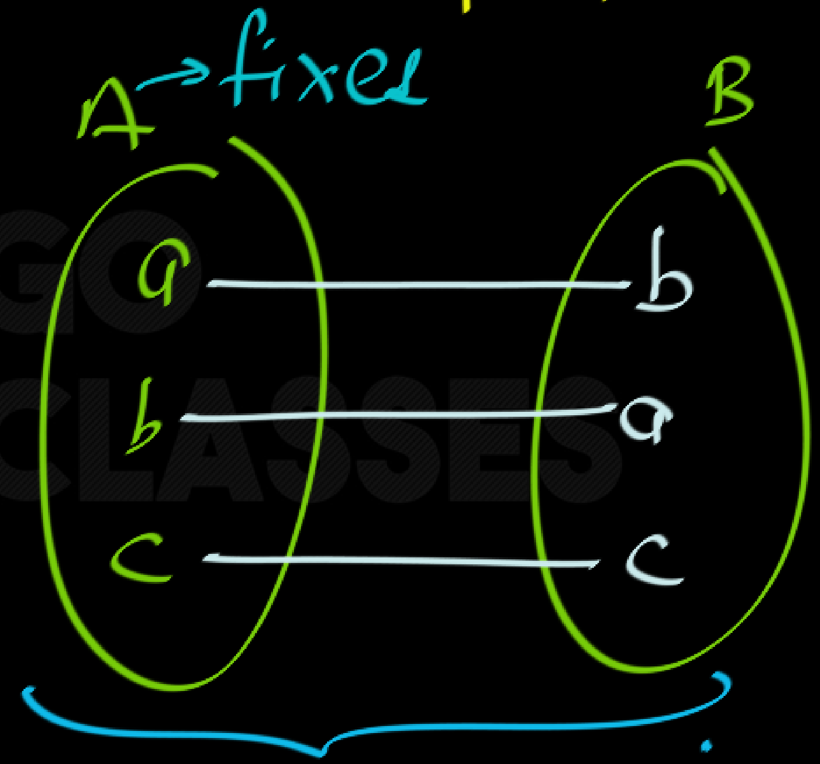


One-to-one Correspondence

Discrete Mathematics

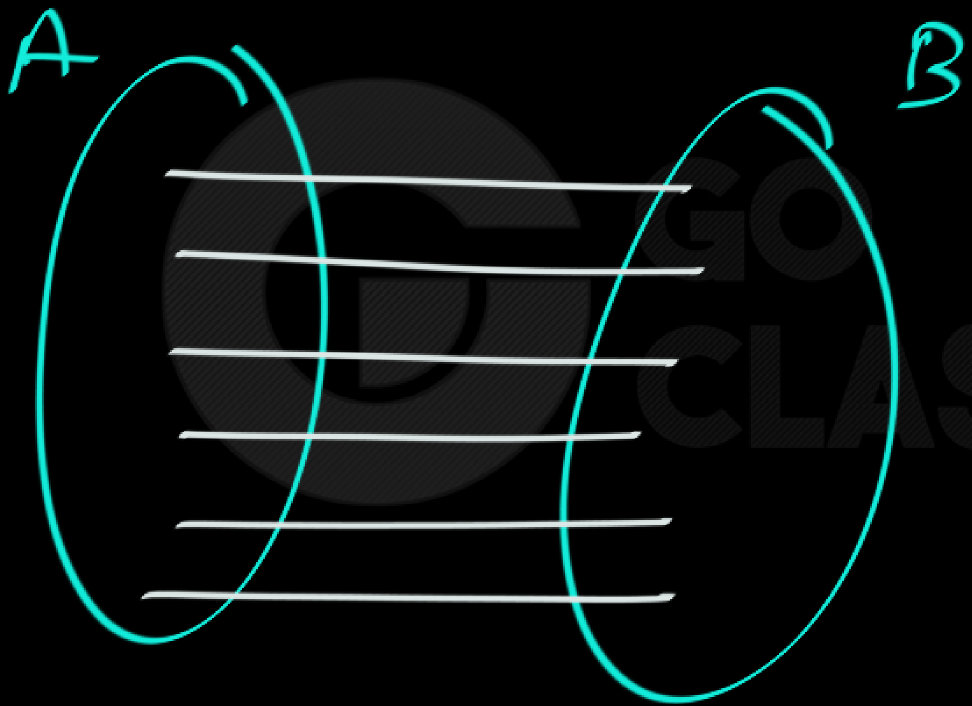
Bijection exists iff |Domain| = |Co-Domain|





Same bijection in beauti-
-ful way.

Bijection ≡ one-one Correspondence



$$\underbrace{|A| = |B|}$$

Q:

f is a function from Set X to Set Y, which is NOT Bijective. Then ??

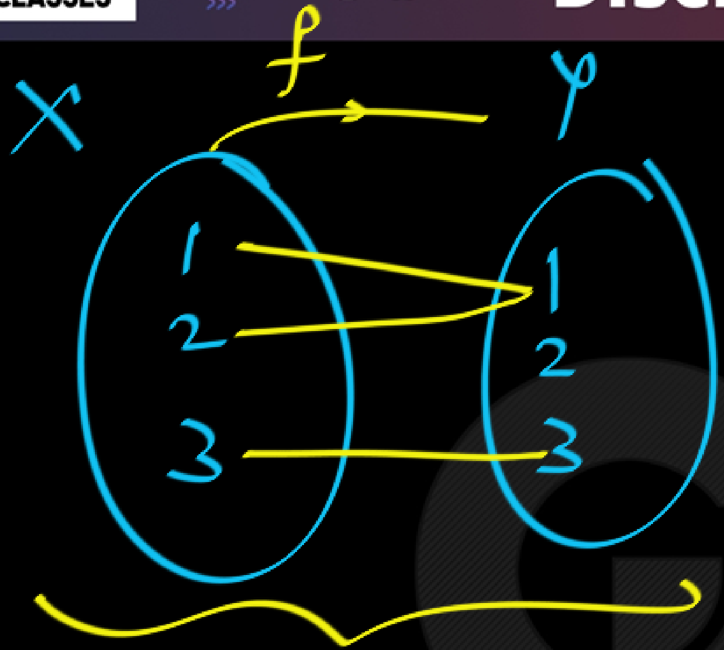
A. $|X| = |Y|$
B. $|X| != |Y|$
C. Can't say anything

Q:

f is a function from Set X to Set Y, which is NOT Bijective. Then ??

A. |X| = |Y|
B. |X| != |Y|
✓ C. Can’t say anything ✓

Discrete Mathematics



NOT bijection

Some bijection Exists.

Bijective Functions & Cardinalities:

Theorem:

Bijective Function EXISTS

iff

|Domain| = |Co-Domain|

Q:

f is a function from Set X to Set Y, which is Bijective. Then ??

A. |X| = |Y|

B. |X| != |Y|

C. Can’t say anything

Q:

f is a function from Set X to Set Y, which is Bijective. Then ??

[✓ A. $|X| = |Y|$]

B. $|X| != |Y|$

C. Can't say anything

Q:

There is NO Bijective function from Set X to Set Y. Then ??

A. |X| = |Y|
B. |X| != |Y|
C. Can’t say anything

Q:

There is NO Bijective function from Set X to Set Y. Then ??

A. $|X| = |Y|$

[✓ B. $|X|$!= $|Y|$]

C. Can't say anything

Q: True / False ?

Given: $|\text{Domain}| = |\text{Co-domain}|$

Then Every function is Bijective.

Q: True / False ?

Given: |Domain| = |Co-domain|

Then Every function is Bijective.

No.

Q: True / False ?

Given: $|\text{Domain}| = |\text{Co-domain}|$

Then Some function is Bijective.

Q: True / False ?

Given: $|\text{Domain}| = |\text{Co-domain}|$

Then Some function is Bijective.

Yes.

Q: True / False ?

Given: |Domain| <= |Co-domain|

Then Every function is Injective.

Q: True / False ?

Given: |Domain| <= |Co-domain|

Then Every function is Injective.

No.

Q: True / False ?

Given: |Domain| <= |Co-domain|

Then Some function is Injective.

Q: True ✓ / False ?

Given: |Domain| <= |Co-domain|

Then Some function is Injective.

Yes.

Q: True / False ?

Given: |Domain| >= |Co-domain|

Then Every function is Surjective.

Q: True / [✓ False] ?

Given: |Domain| >= |Co-domain|

Then Every function is Surjective.

No

Q: True / False ?

Given: |Domain| >= |Co-domain|

Then Some function is Surjective.

Q: True / False ?

Given: |Domain| >= |Co-domain|

Then Some function is Surjective.

Yes.

Intuitive Results about Cardinality of Sets:

✓ 1. *If* $|B| \leq |A|$ *and* $|C| \leq |B|$ *then then* $|C| \leq |A|$.

✓ 2. *If* $|A| = |B|$ *and* $|B| = |C|$ *then* $|A| = |C|$.

Conclusion & Important Results:

Definition: Two sets A and B have the same cardinality ($|A| = |B|$) if and only if there is a bijection from A to B.

Definition: $|A| \leq |B|$ if and only if there is a one-to-one function from A to B.

1. *If* $|B| \leq |A|$ *and* $|C| \leq |B|$ *then then* $|C| \leq |A|$.
2. *If* $|A| = |B|$ *and* $|B| = |C|$ *then* $|A| = |C|$.

finite or infinite

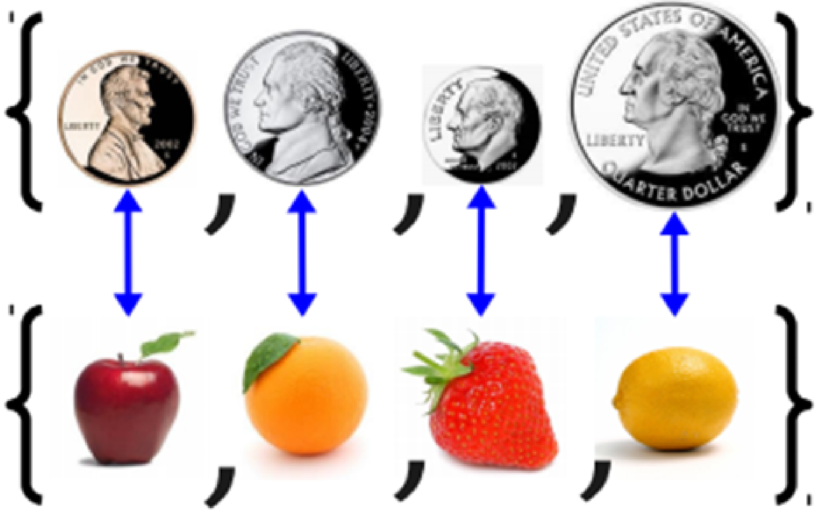
Cardinality

- Recall (from *lecture one!*) that the **cardinality** of a set is the number of elements it contains.
  - Denoted |S|.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200, 300\}| = 3$

Comparing Cardinalities

- The relationships between set cardinalities are defined in terms of functions between those sets.
- $|S| = |T|$ is defined using bijections.

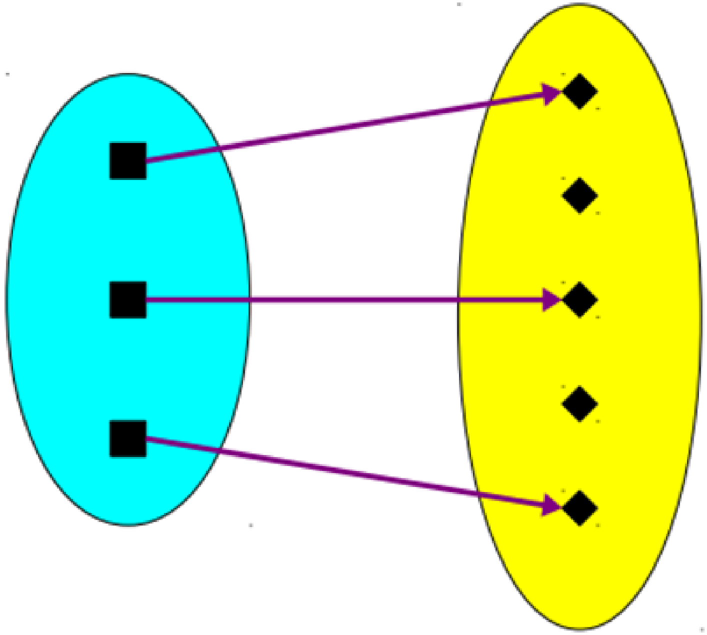
$|S| = |T|$ iff there is a bijection $f : S \rightarrow T$



Comparing Cardinalities

- We define $|S| \leq |T|$ as follows:

$|S| \leq |T|$ iff there is an injection $f : S \rightarrow T$



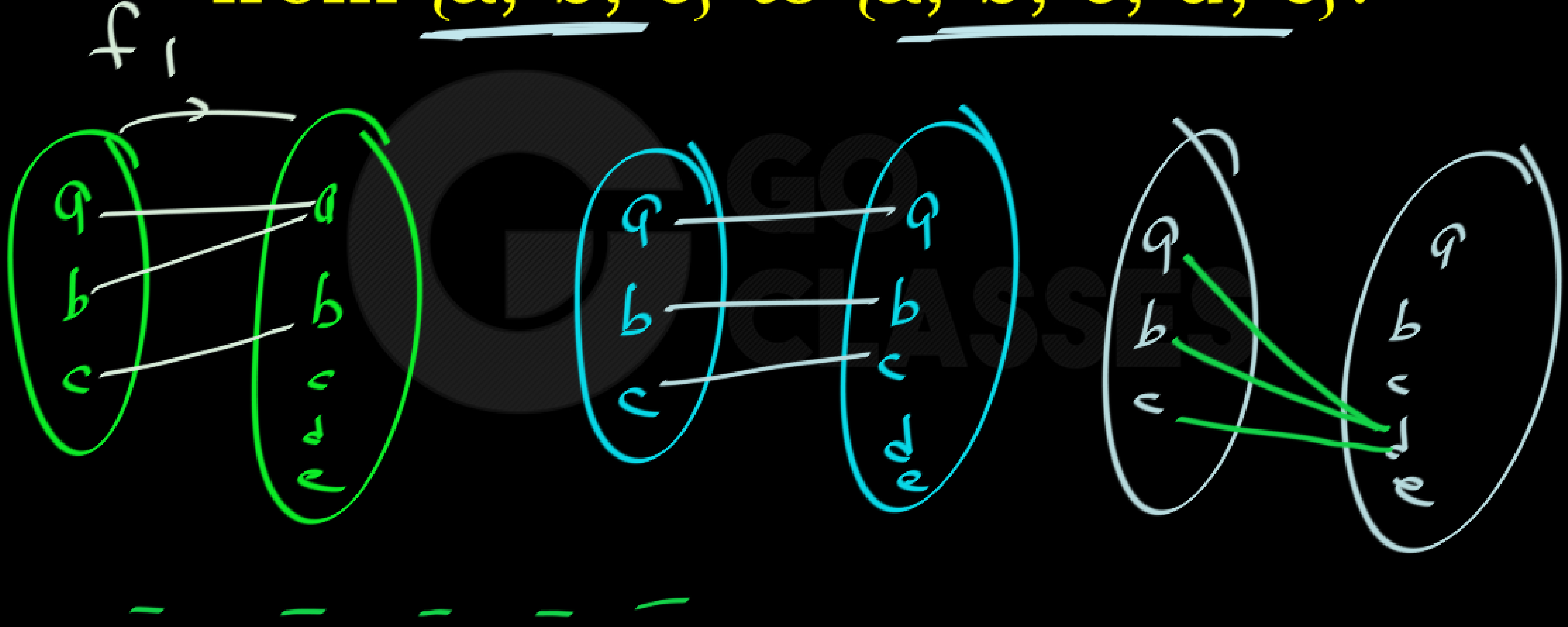
If a function is one-to-one, then $| A | \leq | B |$.

Counting Functions

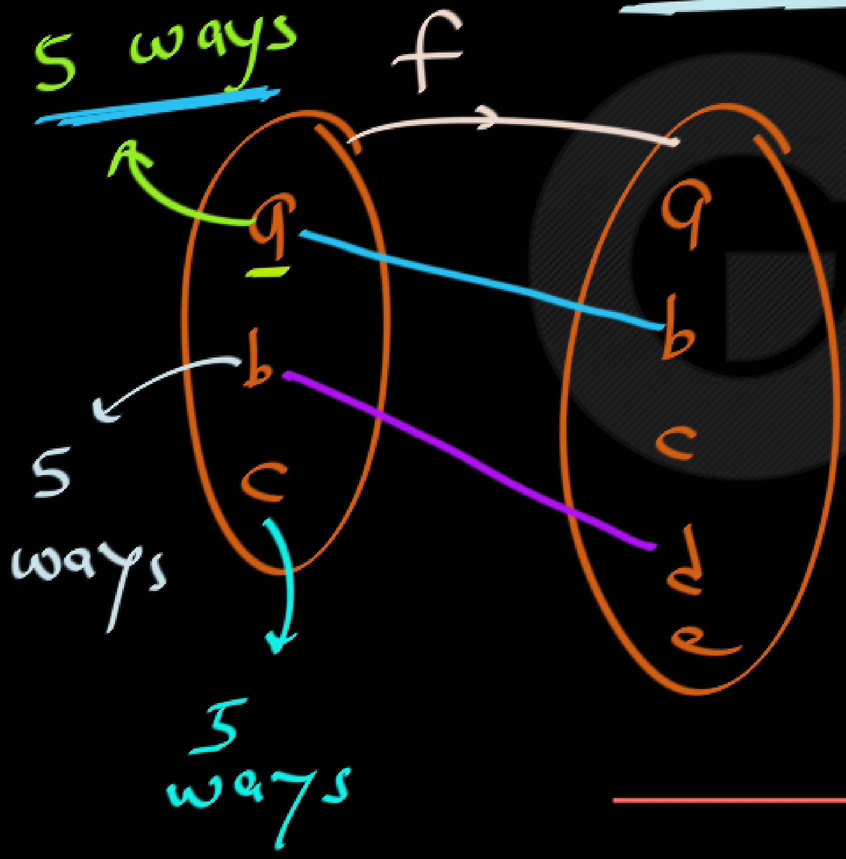
Counting
the total number of Functions

How many functions are there from {a, b, c} to {a, b, c, d, e}?

How many functions are there from {a, b, c} to {a, b, c, d, e}?



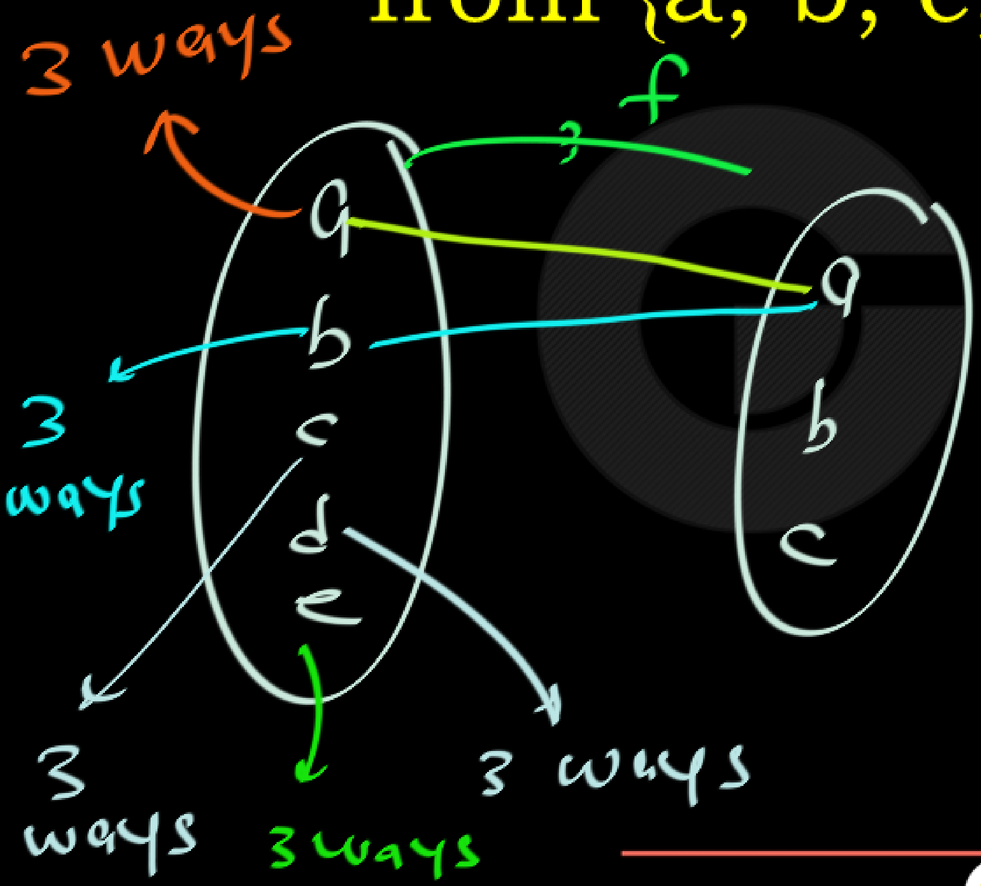
How many functions are there from {a, b, c} to {a, b, c, d, e}?



$$\# \text{functions} = 5 \times 5 \times 5 = 5^{3} = |\text{Co-Domain}|^{|\text{Domain}|}$$

How many functions are there from {a, b, c, d, e} to {a, b, c}?

How many functions are there from {a, b, c, d, e} to {a, b, c}?



$$\#\text{functions} = 3 \times 3 \times 3 \times 3 \times 3 = 3^5 = |\text{Co-Domain}|^{|\text{Domain}|}$$

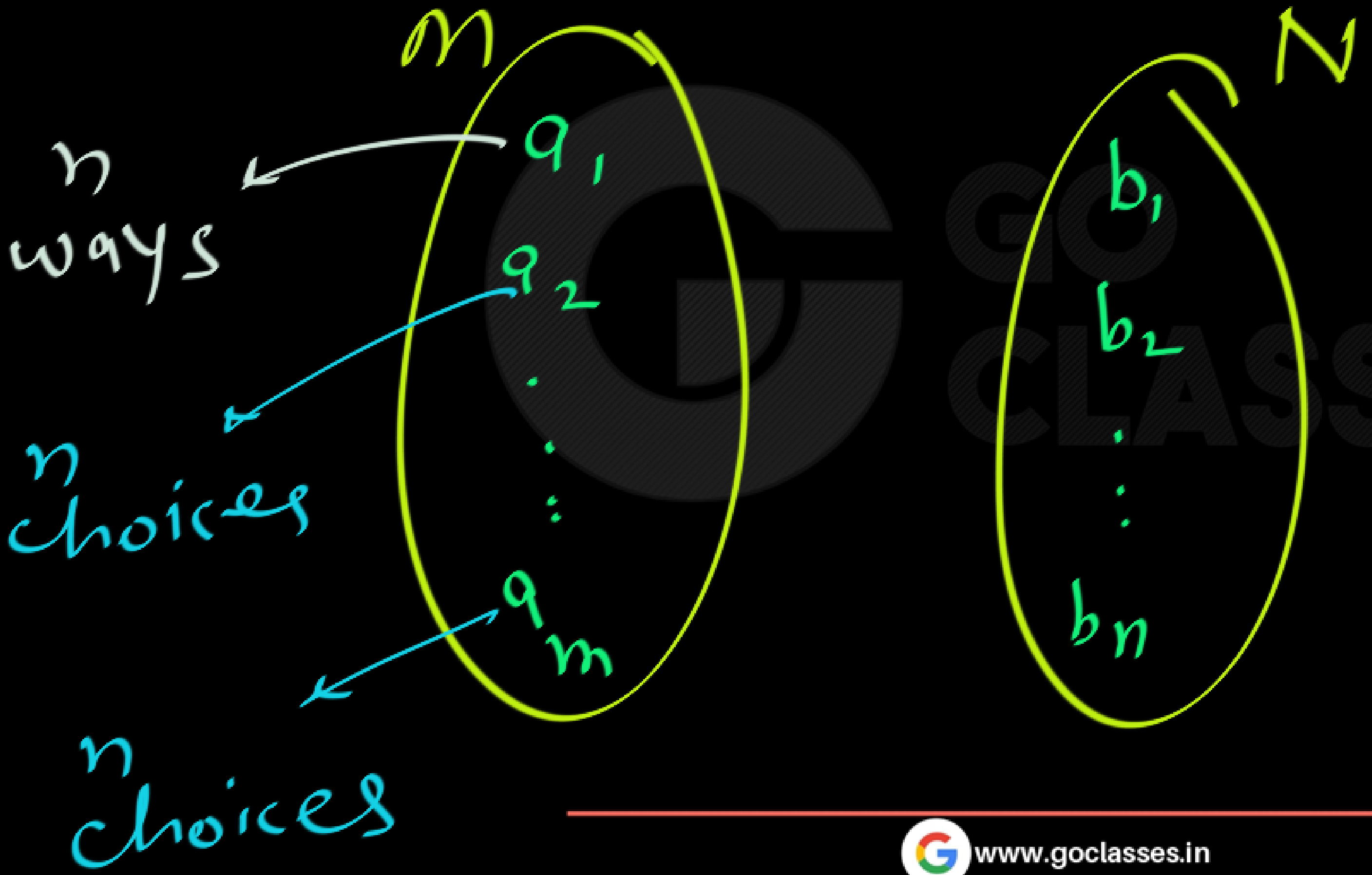
Counting the number of Functions:

Let M and N be finite sets with cardinalities $|M| = m$ and $|N| = n$.

number of functions $M \longrightarrow N$:

$$= |\text{Co-Domain}|^{|\text{Domain}|} = n^{m}$$

$f : M \rightarrow N \quad ; \quad |M| = m \quad ; \quad |N| = n$



$\#\text{functions} = n \times n \times \cdots \times n$

$= n^m$ ✓

Counting Injective Functions

How many Injective functions are there from {a, b, c, d, e} to {a, b, c}?

How many Injective functions are there from {a, b, c, d, e} to {a, b, c}?

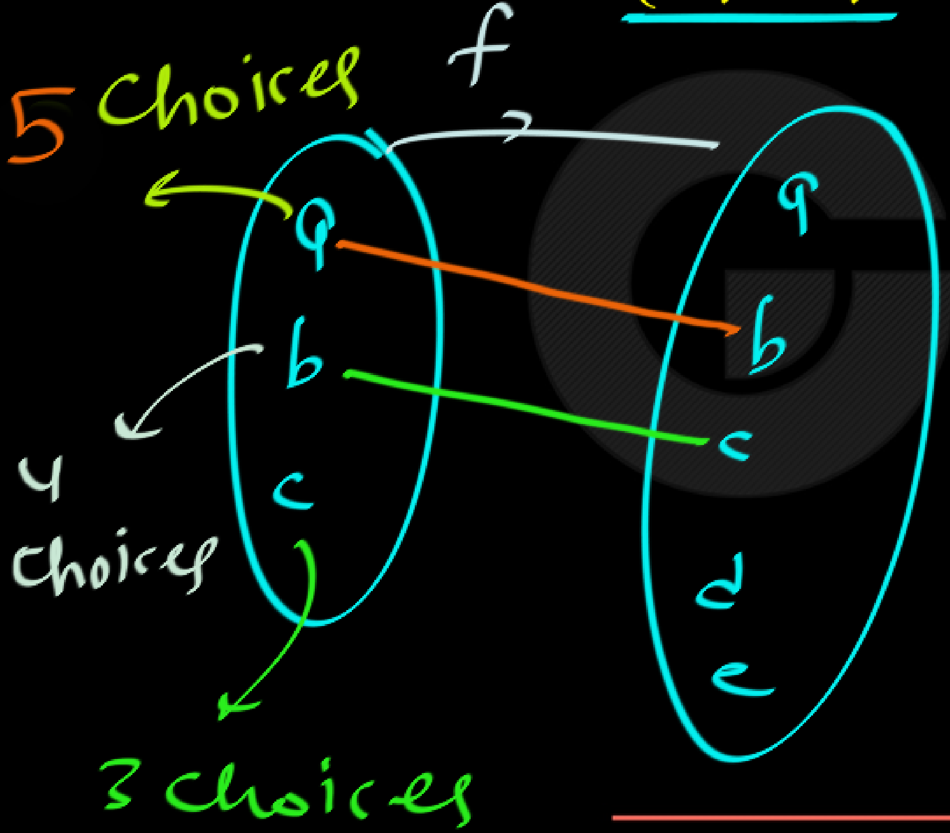
0

Injective function exists iff

|Domain| ≤ |Co-Domain|

How many Injective functions are there from {a, b, c} to {a, b, c, d, e}?

How many Injective functions are there from {a, b, c} to {a, b, c, d, e}?



1-1 functions

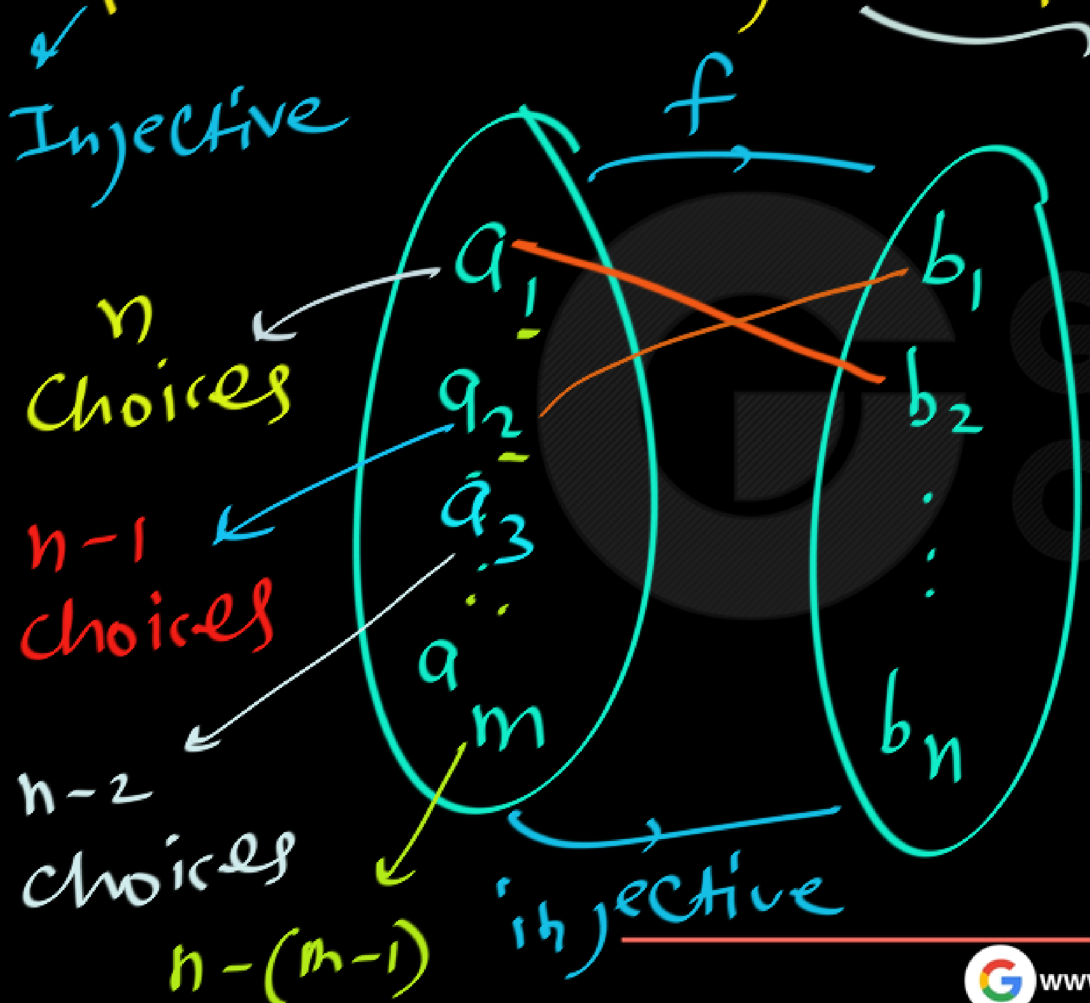
$$= \underbrace{5 \times 4 \times 3}$$

Discrete Mathematics

$f : M \to N$; $\underbrace{|M| \le |N|}$

$|M| = m$

$|N| = n$



\# one-one function

$$= \underbrace{(n)(n-1)}(n-2)\cdots(n-m+1)$$

Counting Injective Functions:

Let M and N be finite sets with cardinalities $|M| = m$ and $|N| = n$.

Number of Injective functions $M \to N$

$$\begin{cases} ① \text{ if } \underbrace{|m| > |N|}_{} \text{ then } 0. \\ ② \text{ if } |m| \le |N| \text{ then } (n)(n-1)\cdots(n-m+1) \end{cases}$$

In Combinatorics

$$(n)(n-1)\cdots(n-m+1) = {}^{n}P_{m} = \frac{n!}{(n-m)!}$$

Counting Bijective Functions

How many Bijective functions are there from $\{a, b, c, d, e\}$ to $\{a, b, c\}$?

How many Bijective functions are there from {a, b, c, d, e} to {a, b, c}?

0

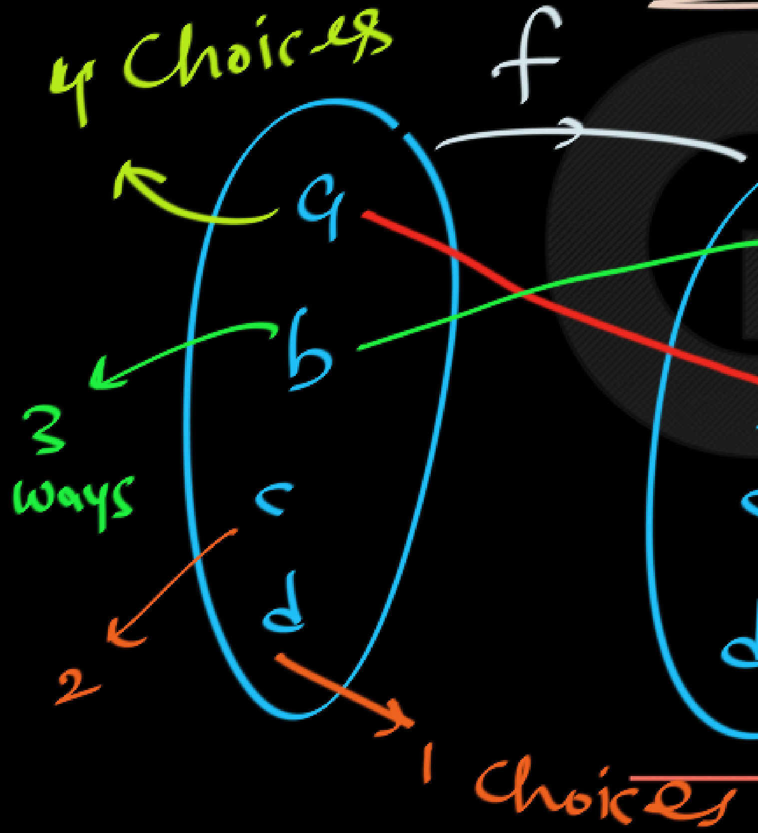
Bijective function exists iff

$$\underbrace{|\text{Domain}| = |\text{Co-Domain}|}$$

How many Bijective functions are there from {a, b, c} to {a, b, c, d, e}?

0

How many Bijective functions are there from {a, b, c, d} to {a, b, c, d}?



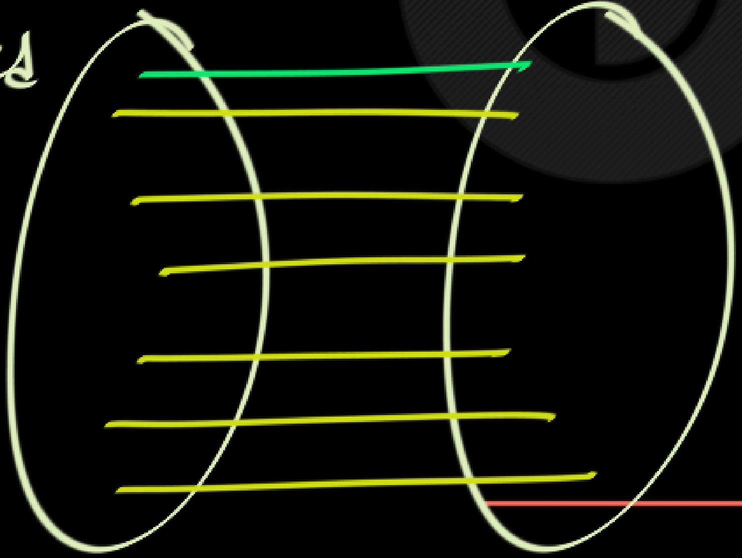
$$\#\text{bijections} = \underbrace{4\times3\times2\times1}_{} = \boxed{4!}$$

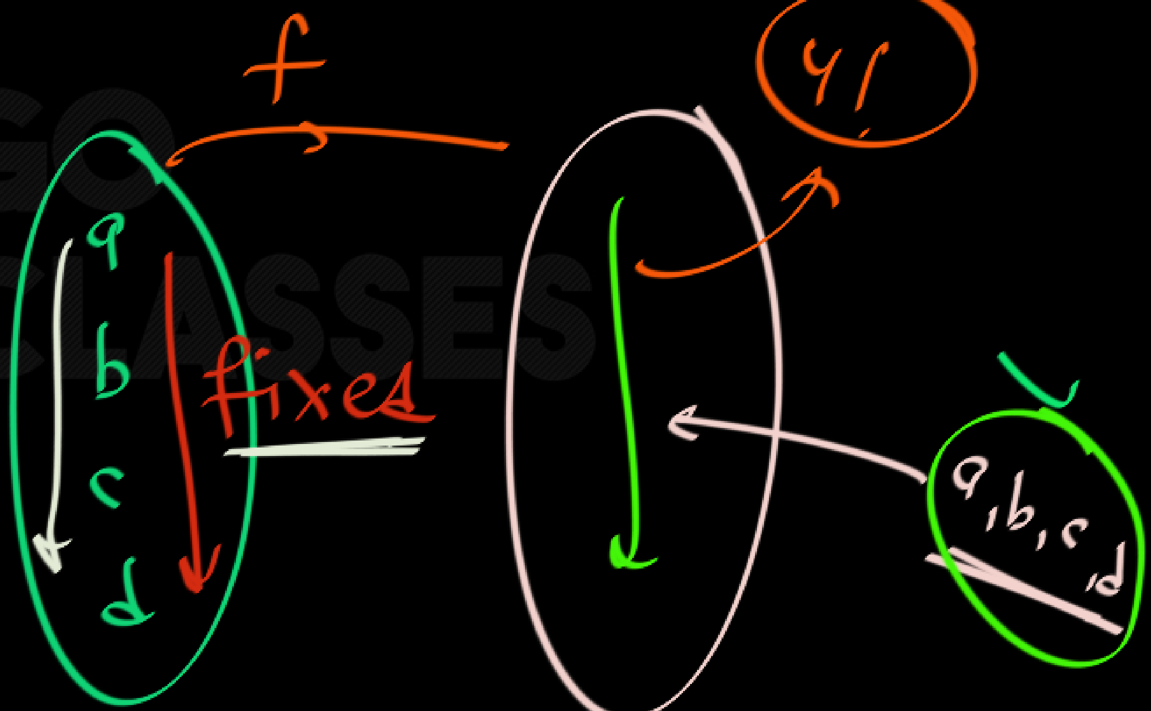
4 factorial

How many Bijective functions are there from {a, b, c, d} to {a, b, c, d}?

Bijection

fixes





Counting Bijective Functions:

Let M and N be finite sets with cardinalities $|M| = m$ and $|N| = n$.

$f : M \to N$

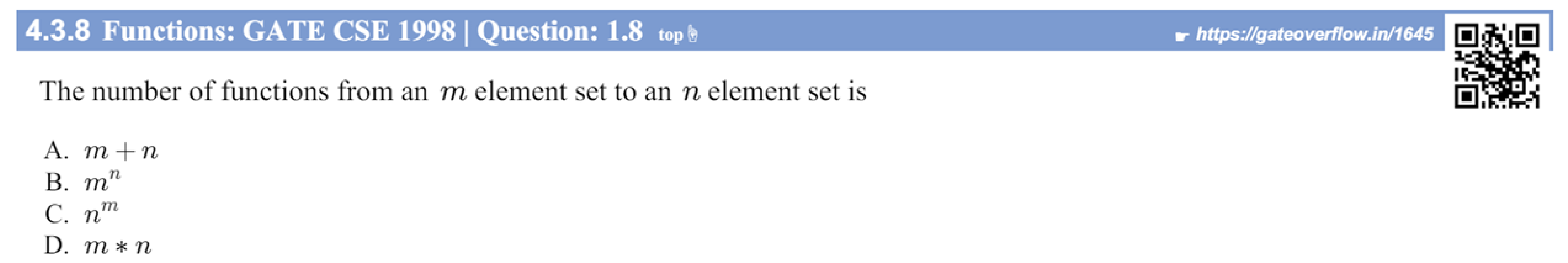
$$\text{number of } \underline{\text{bijections}} = \begin{cases} 0 & ; \; m > n \\ 0 & ; \; m < n \\ m! & ; \; \underline{m = n} \end{cases}$$

1.11. How many injective functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$?

Solution. Let f be such a function. Then $f(1)$ can take 5 values, $f(2)$ can then take only 4 values and $f(3)$ - only 3. Hence the total number of functions is $5 \times 4 \times 3 = 60$.

Counting Surjective Functions:

We study in the Combinatorics Module.



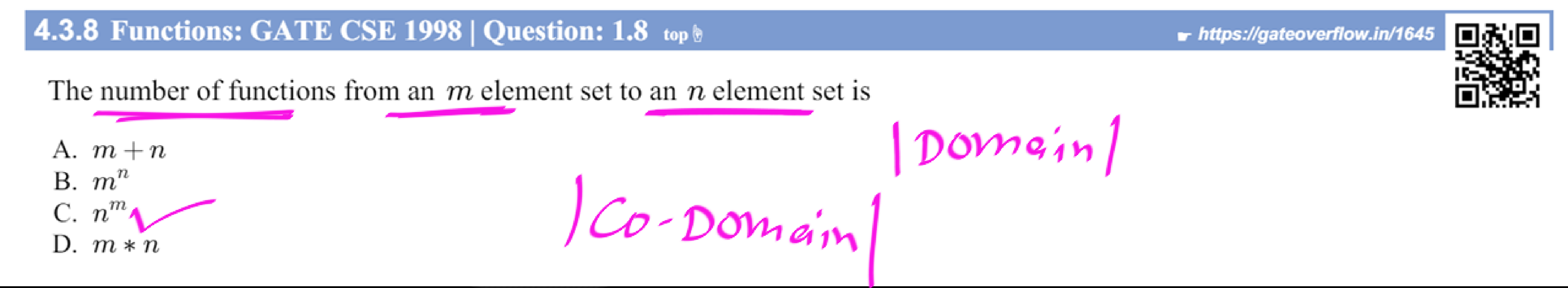
4.3.8 Functions: GATE CSE 1998 | Question: 1.8 top

https://gateoverflow.in/1645

The number of functions from an m element set to an n element set is

A. $m+n$
B. m^n
C. n^m
D. $m*n$

Discrete Mathematics



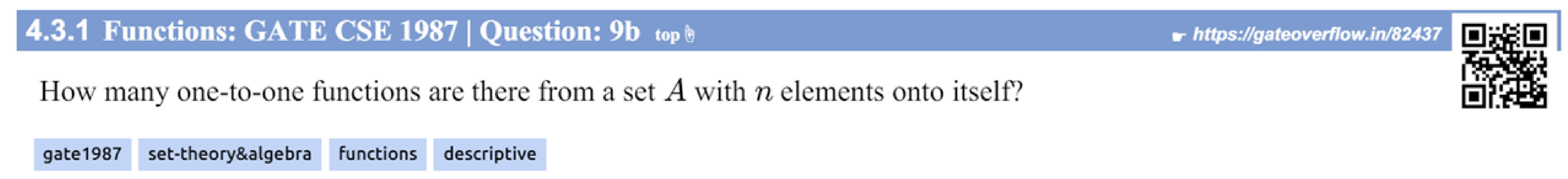
4.3.8 Functions: GATE CSE 1998 | Question: 1.8 top

https://gateoverflow.in/1645

The number of functions from an m element set to an n element set is

A. $m + n$
B. m^n
C. n^m ✓
D. $m * n$

$$|\text{Co-Domain}|^{|\text{Domain}|}$$



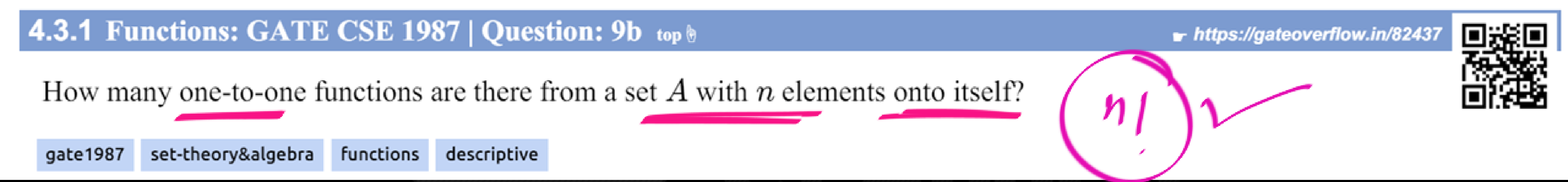
4.3.1 Functions: GATE CSE 1987 | Question: 9b top

☛ *https://gateoverflow.in/82437*

How many one-to-one functions are there from a set A with n elements onto itself?

gate1987 set-theory&algebra functions descriptive

Discrete Mathematics



4.3.1 Functions: GATE CSE 1987 | Question: 9b top

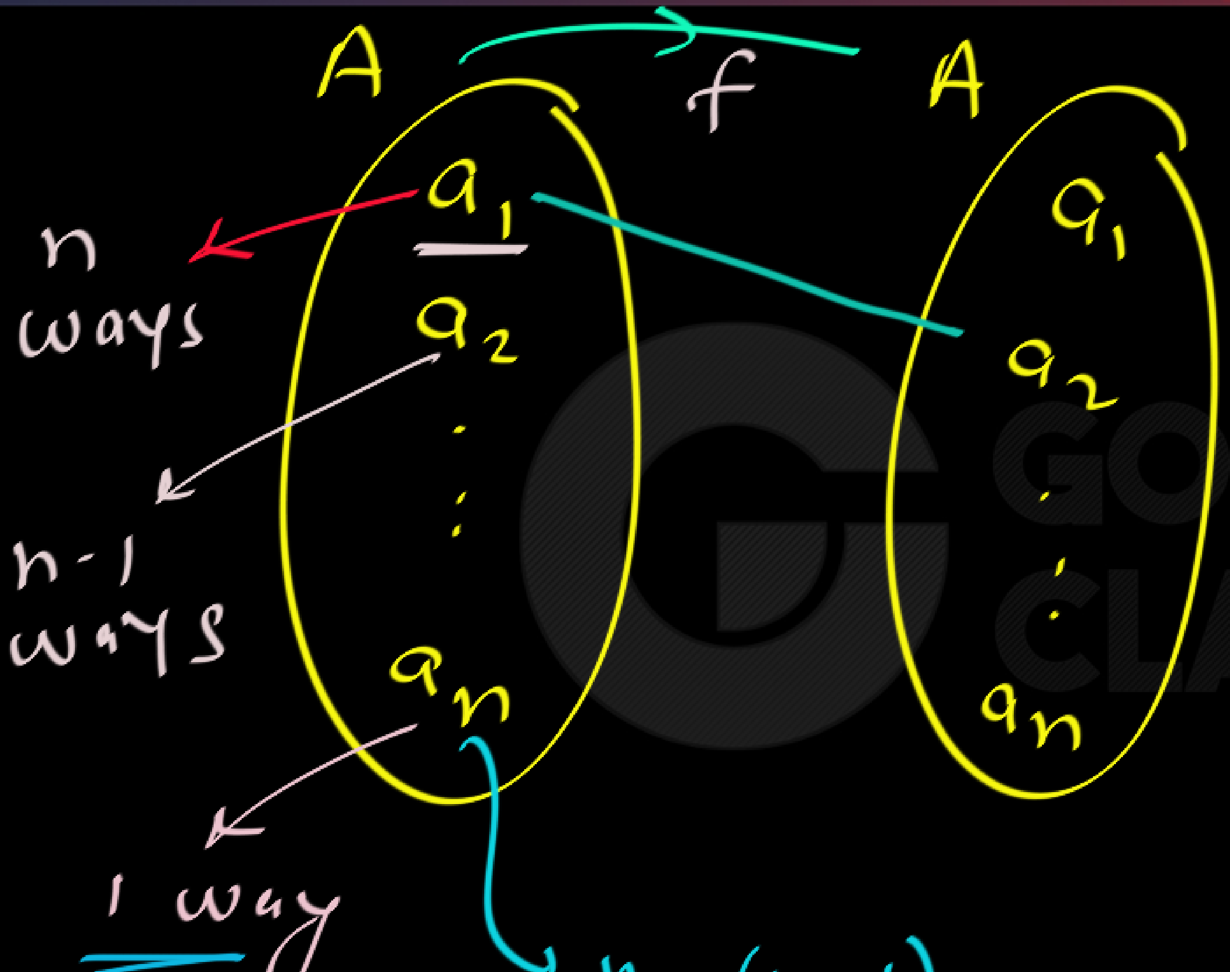
https://gateoverflow.in/82437

How many one-to-one functions are there from a set A with n elements onto itself?

$\boxed{n!}$ ✓

gate1987 set-theory&algebra functions descriptive

Discrete Mathematics



Injective functions from A to A:

$(n)(n-1)(n-2)\ldots 1$

$= n!$

n-(m-1) ways but m=n now

Let $f(n) = 2$ for $n \leq 2$ and $f(n) = f(n-1) + f(n-2) + 1$. Compute the values of $f(n)$ for $n \leq 6$.

Let $f(n) = 2$ for $n \leq 2$ and $f(n) = f(n-1) + f(n-2) + 1$. Compute the values of $f(n)$ for $n \leq 6$.

$f(1) = 2$

$f(2) = 2$

$f(n) = f(n-1) + f(n-2) + 1$

$f(3) = ? = f(2) + f(1) + 1 = 2 + 2 + 1 = 5$

$f(4) = f(3) + f(2) + 1 = 5 + 2 + 1 = 8$

$f(5) = 8 + 5 + 1 = 14$; $f(6) = 14 + 8 + 1 = 23$

Let $f(n) = 1$ for $n \leq 2$ and $f(n) = 2f(n-1) - 1$. Compute the values of $f(n)$ for $n \leq 5$.

Let $f(n) = 1$ for $n \leq 2$ and $f(n) = 2f(n-1) - 1$. Compute the values of $f(n)$ for $n \leq 5$.

All $f(i) = 1$ ✓

$f(1) = 1$

$f(2) = 1$

$f(n) = 2f(n-1) - 1$

$f(\underset{n}{3}) = 2f(2) - 1 = 2 \times 1 - 1 = 1$

$f(4) = 2f(3) - 1 = 1$;

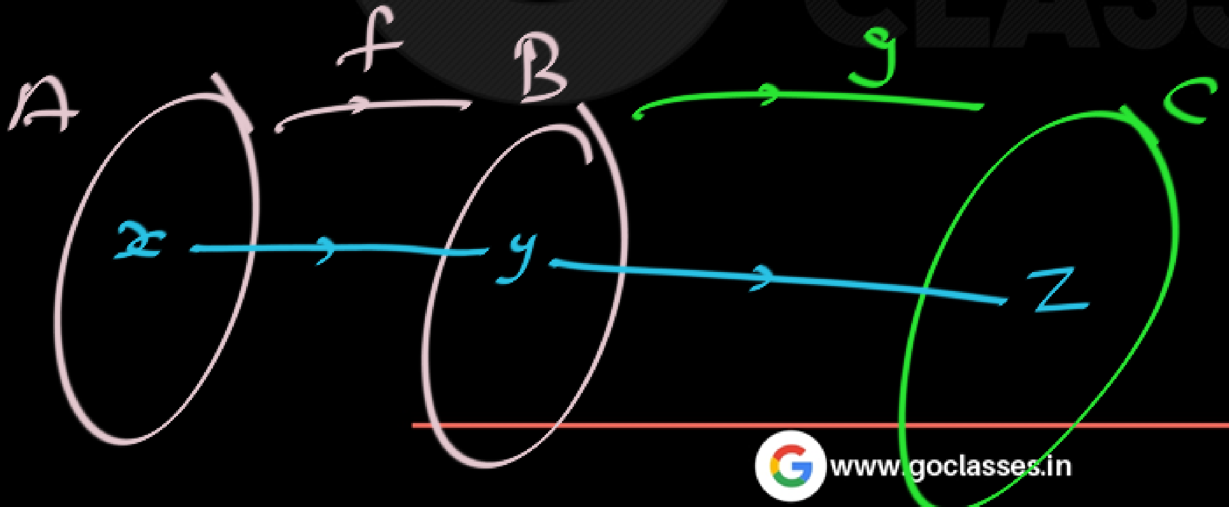
$f(5) = 2f(4) - 1 = 1$

Functions Composition

function Composition:

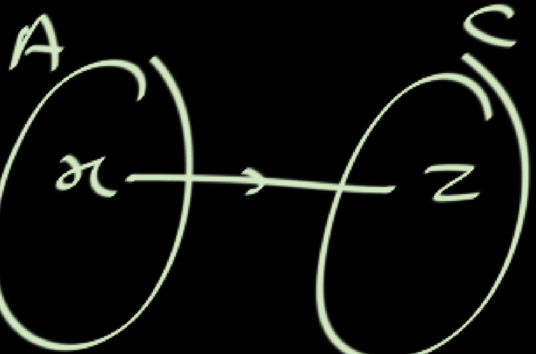
operation to create NEW function.

$f: A \to B$; $g: B \to C$



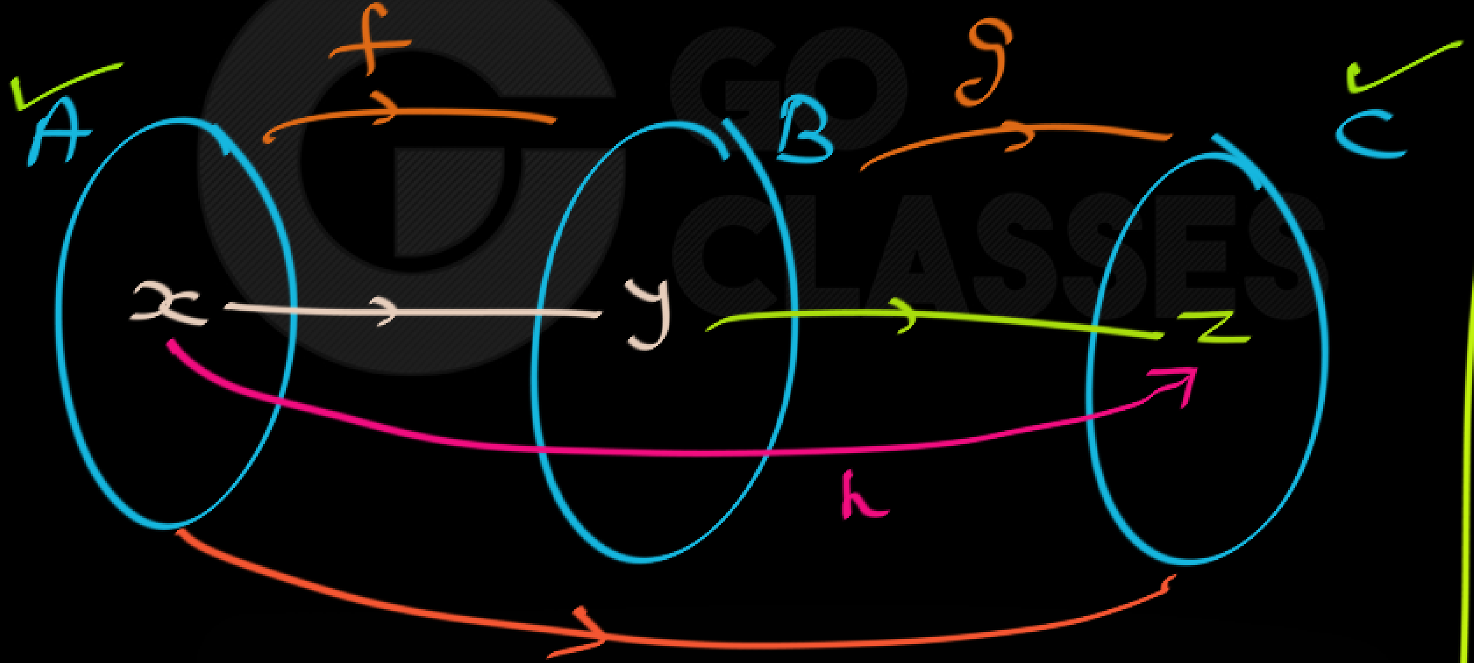
New fun.

from f, g :



Composition creates new function Transitively.

$f : A \to B \; ; \; g : B \to C$



new function $h : A \to C$

$h(x) = z$

$g(f(x))$

$= z$

$h = g \circ f$

Compositions

$h = \underline{g \circ f}$

new function

h

$h(x) = g(f(x))$

www.microsoft.com

www.apple.com

www.google.com

www.microsoft.com

www.apple.com

www.google.com

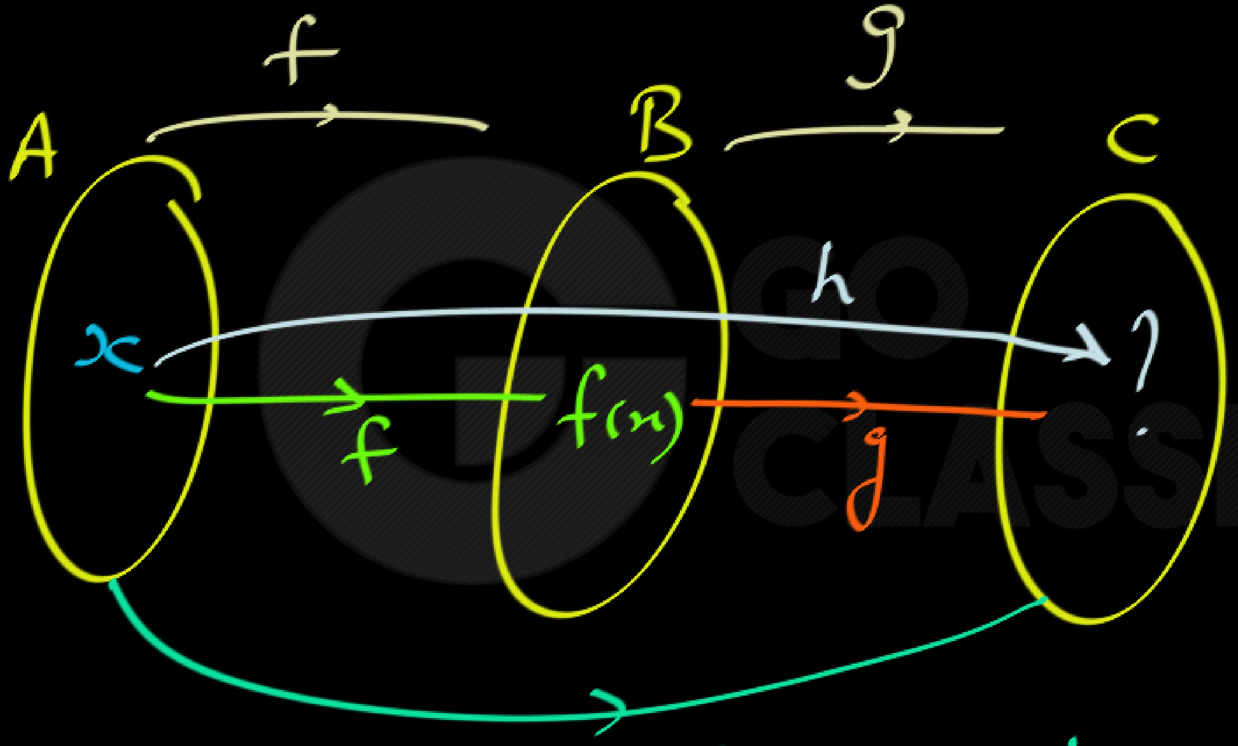
$g \circ f$

$g \circ f$

$g \circ f$

$f : A \rightarrow B \; ; \; g : B \rightarrow C$

$h(x) = g(f(x))$



$\underline{h = g \circ f}$

new function $h : \underbrace{A \rightarrow C}$

Domain ↑ (A)

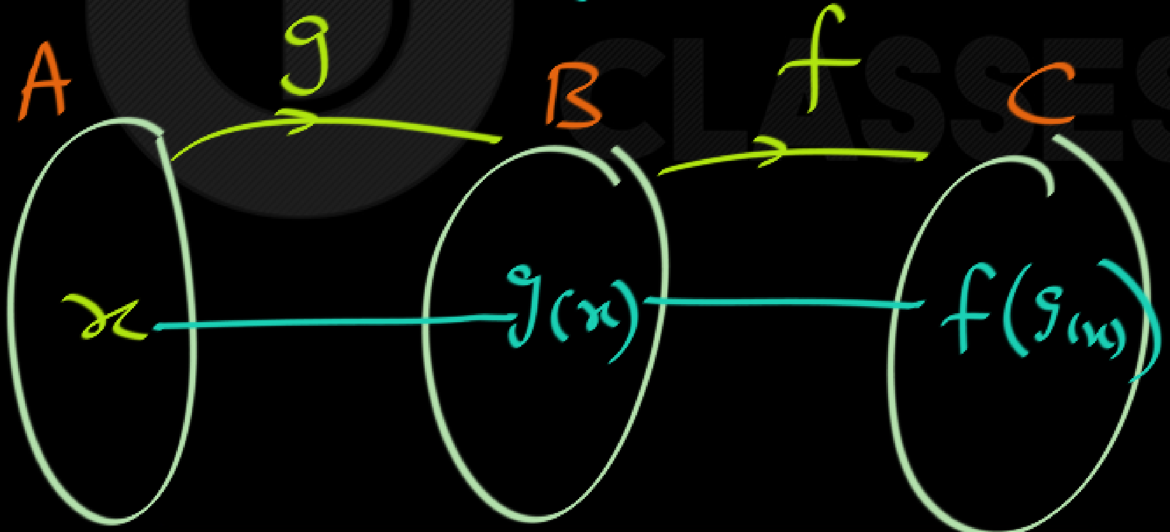
Codomain ↑ (C)

$$g \circ f = g(f(x))$$

Composition operation

$$g \circ f(x) = g(f(x))$$

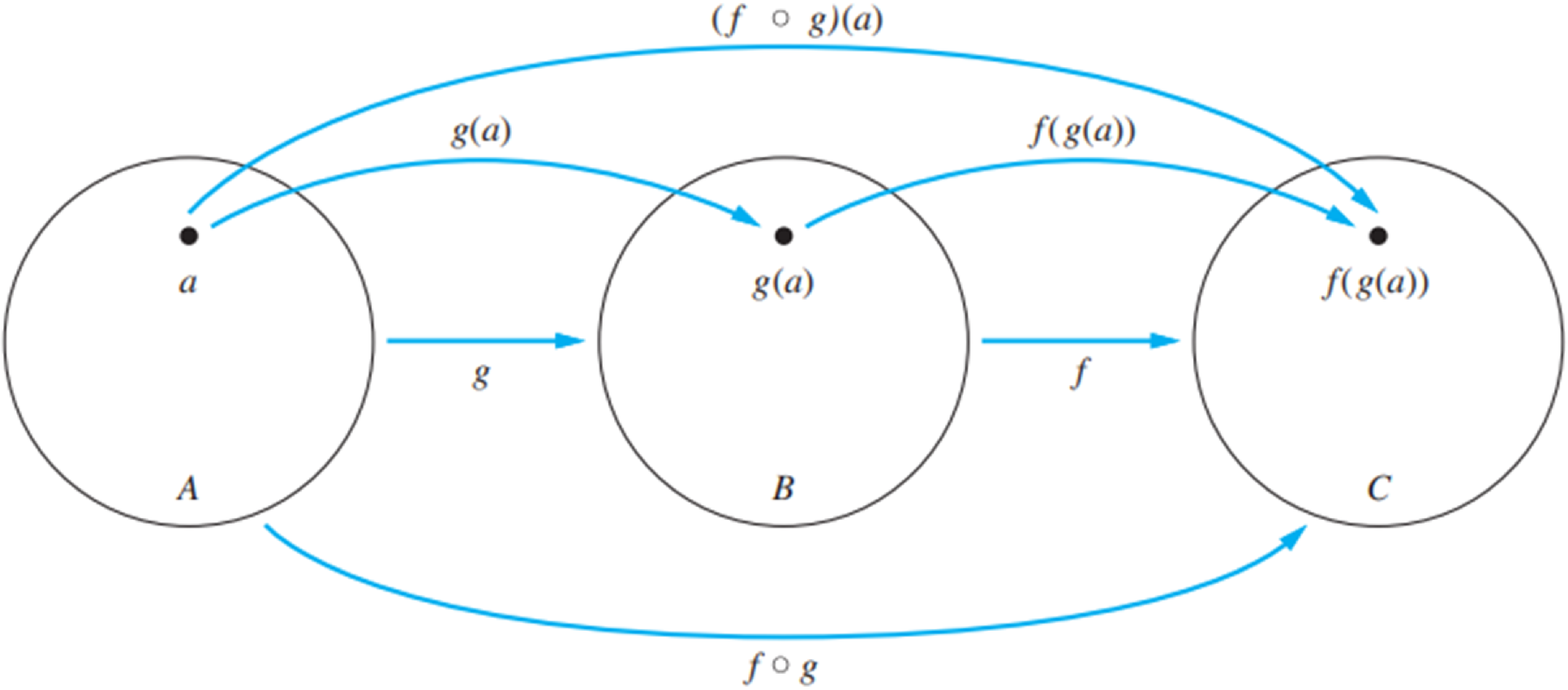
To Define $f \circ g$: $= f(g(x))$



$g : A \rightarrow B$

$f : B \rightarrow C$

$$\left\{ \begin{array}{l} f \circ g(x) = f(g(x)) \\ g \circ f(x) = g(f(x)) \end{array} \right\}$$


$(f \circ g)(a)$
$g(a)$
$f(g(a))$
a
$g(a)$
$f(g(a))$
g
f
A
B
C
$f \circ g$

Function Composition

- Let $f : A \to B$ and $g : B \to C$.
- The **composition of f and g** (denoted **$g \circ f$**) is the function $g \circ f : A \to C$ defined as

$$(g \circ f)(x) = g(f(x))$$

- Note that f is applied first, but f is on the right side!
- Function composition is **associative**:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\underbrace{f \circ (g \circ h) = (f \circ g) \circ h}$$

function Composition is Associative.

$f : A \to A \; ; \quad g : A \to A$

$f(x) = 2x \; ; \quad g(x) = x + 1$

$f \circ g(x) = f(g(x)) = f(x+1)$

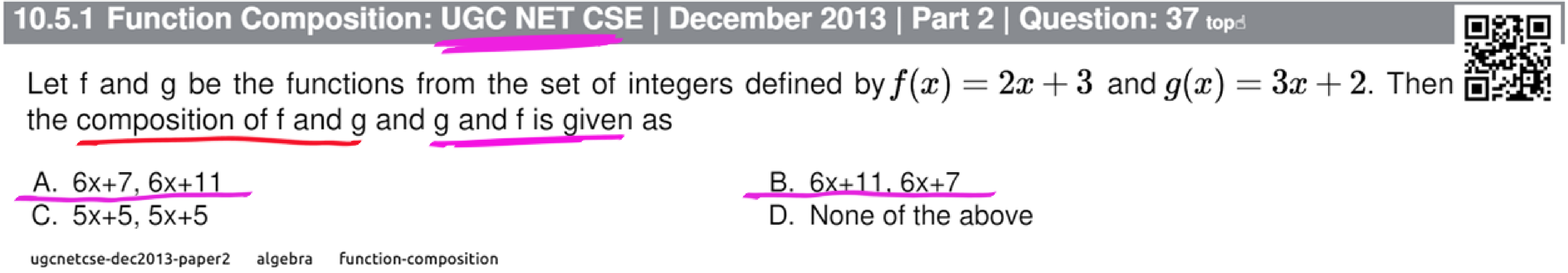
$= 2(x+1) = \underline{2x+2}$

$g \circ f(x) = g(f(x)) = g(2x) = \underline{2x+1}$

Function Composition

- Suppose $f : A \to A$ and $g : A \to A$.
- Then both $g \circ f$ and $f \circ g$ are defined.
- Does $g \circ f = f \circ g$? → Not Necessarily
- **In general, no:**
    - Let $f(x) = 2x$
    - Let $g(x) = x + 1$
    - $(g \circ f)(x) = g(f(x)) = g(2x) = 2x + 1$
    - $(f \circ g)(x) = f(g(x)) = f(x + 1) = 2x + 2$

$$\left.\begin{array}{l} fog(x) = 6x+7 \\ gof(x) = 6x+11 \end{array}\right\}$$



10.5.1 Function Composition: UGC NET CSE | December 2013 | Part 2 | Question: 37 top

Let f and g be the functions from the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. Then the composition of f and g and g and f is given as

A. 6x+7, 6x+11

B. 6x+11, 6x+7

C. 5x+5, 5x+5

D. None of the above

ugcnetcse-dec2013-paper2 algebra function-composition

Composition of f and g $\Rightarrow$ gof or fog ??

" " g and f

$f : Z \to Z \; ; \; g : Z \to Z$

$f(x) = 2x + 3 \; ; \; g(x) = 3x + 2$

$$f \circ g(x) = f(g(x)) = f(3x+2) = 2(3x+2) + 3$$

$$= 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(2x+3) = 3(2x+3) + 2$$

$$= 6x + 11$$

A Small Misconception/Confusion:

What is Composition of $\underbrace{f \text{ and } g}$??

$$\left.\begin{array}{l} g \circ f \\ f \circ g \end{array}\right\}$$ which is Correct?

Discrete_Mathematics_and_Its_Applications_(7th_Edition).pdf 168 / 1071 150%

Keneth Rosen

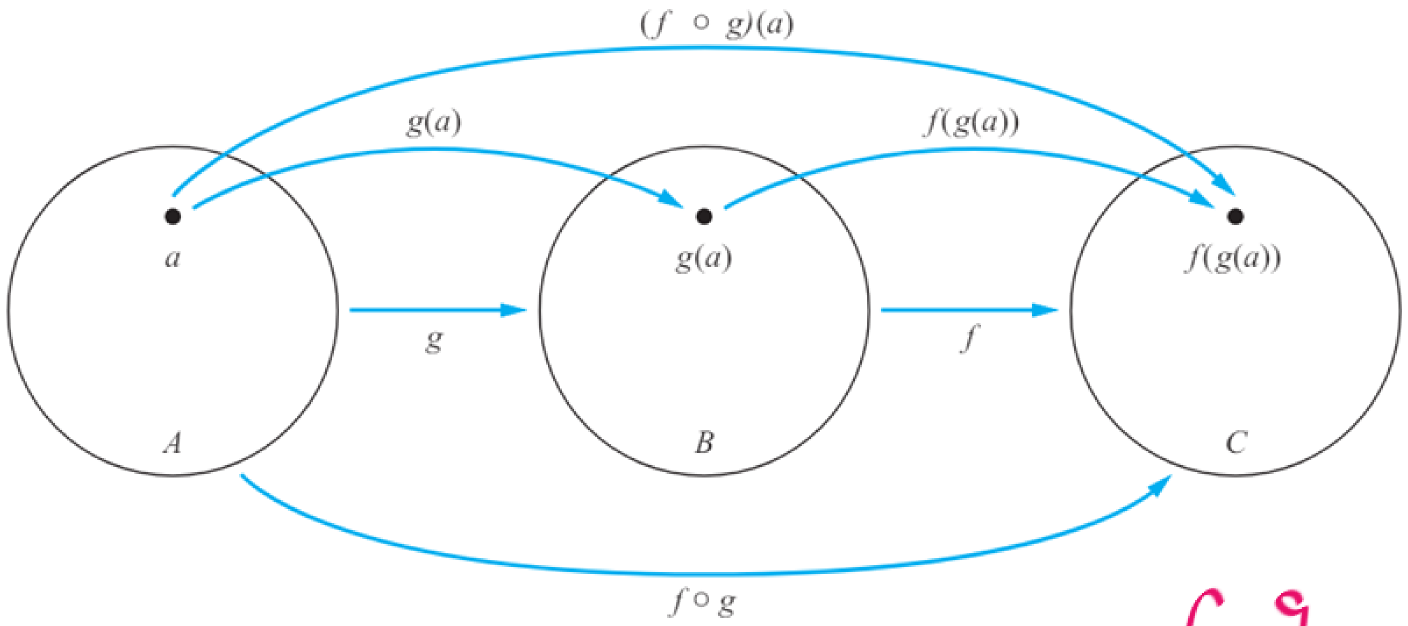
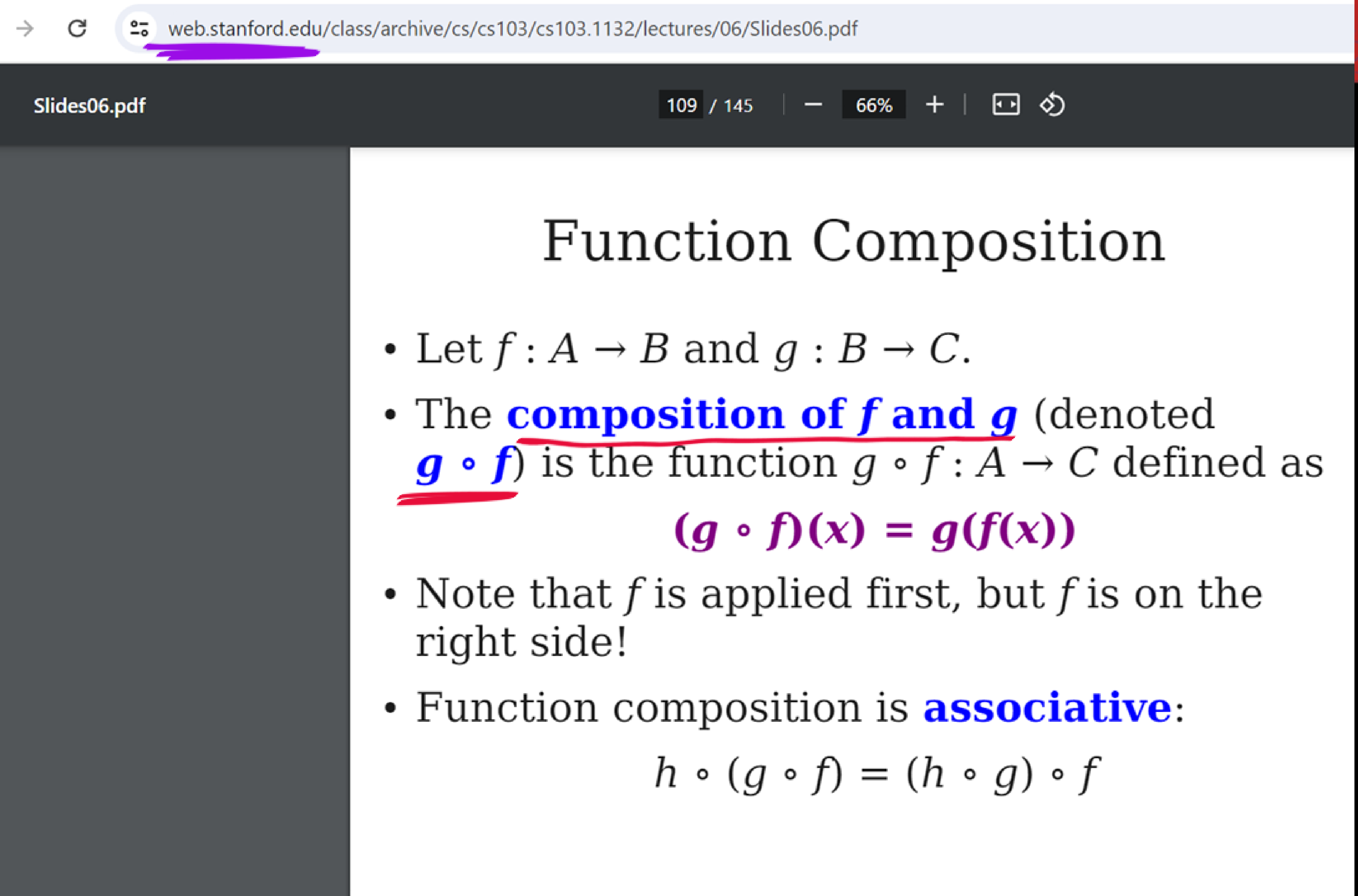


FIGURE 7 **The Composition of the Functions *f* and *g*.** ≡ fog



Function Composition

- Let $f : A \to B$ and $g : B \to C$.
- The **composition of f and g** (denoted **$g \circ f$**) is the function $g \circ f : A \to C$ defined as

$$\mathbf{(g \circ f)(x) = g(f(x))}$$

- Note that f is applied first, but f is on the right side!
- Function composition is **associative**:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Composition of f and g :

Ambiquous

$f \circ g$

$g \circ f$

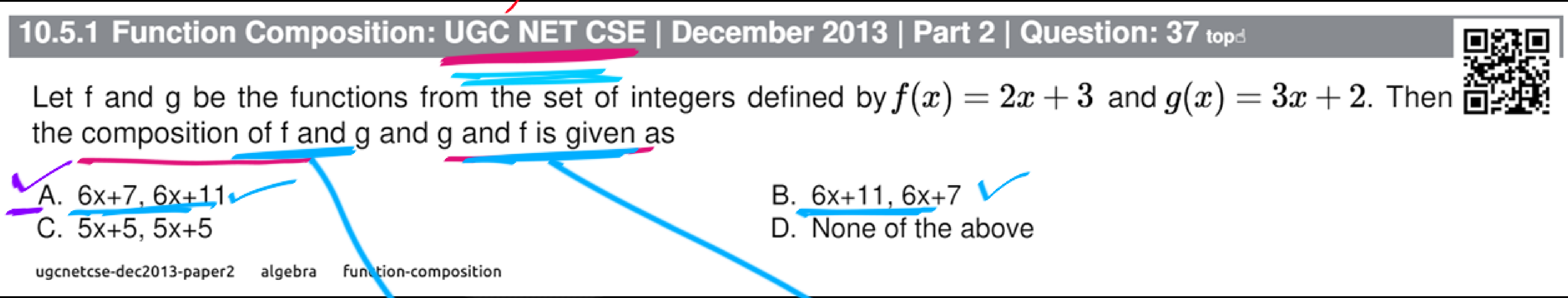
Depends on Author

fog ✓

gof ✓

Discrete Mathematics

DUMB



10.5.1 Function Composition: UGC NET CSE | December 2013 | Part 2 | Question: 37 top

Let f and g be the functions from the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. Then the composition of f and g and g and f is given as

✓ A. 6x+7, 6x+11 ✓

B. 6x+11, 6x+7 ✓

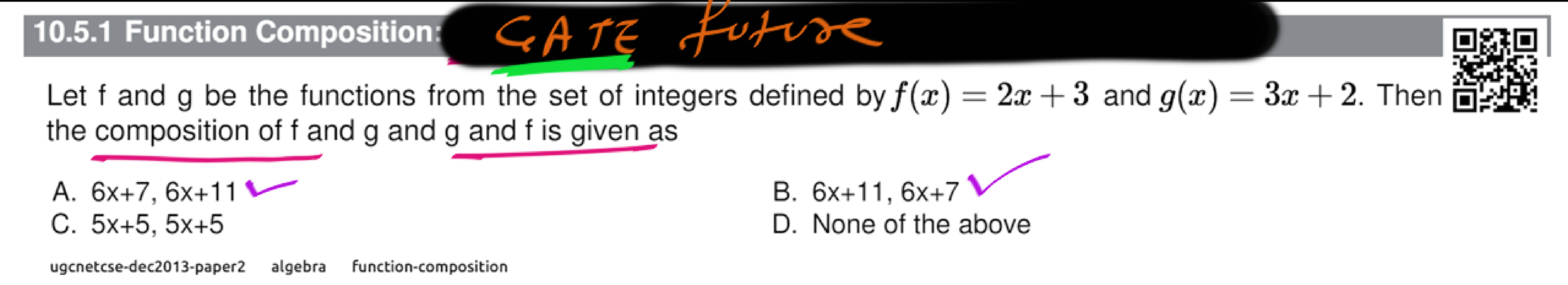
C. 5x+5, 5x+5

D. None of the above

ugcnetcse-dec2013-paper2 algebra function-composition

fog

gof



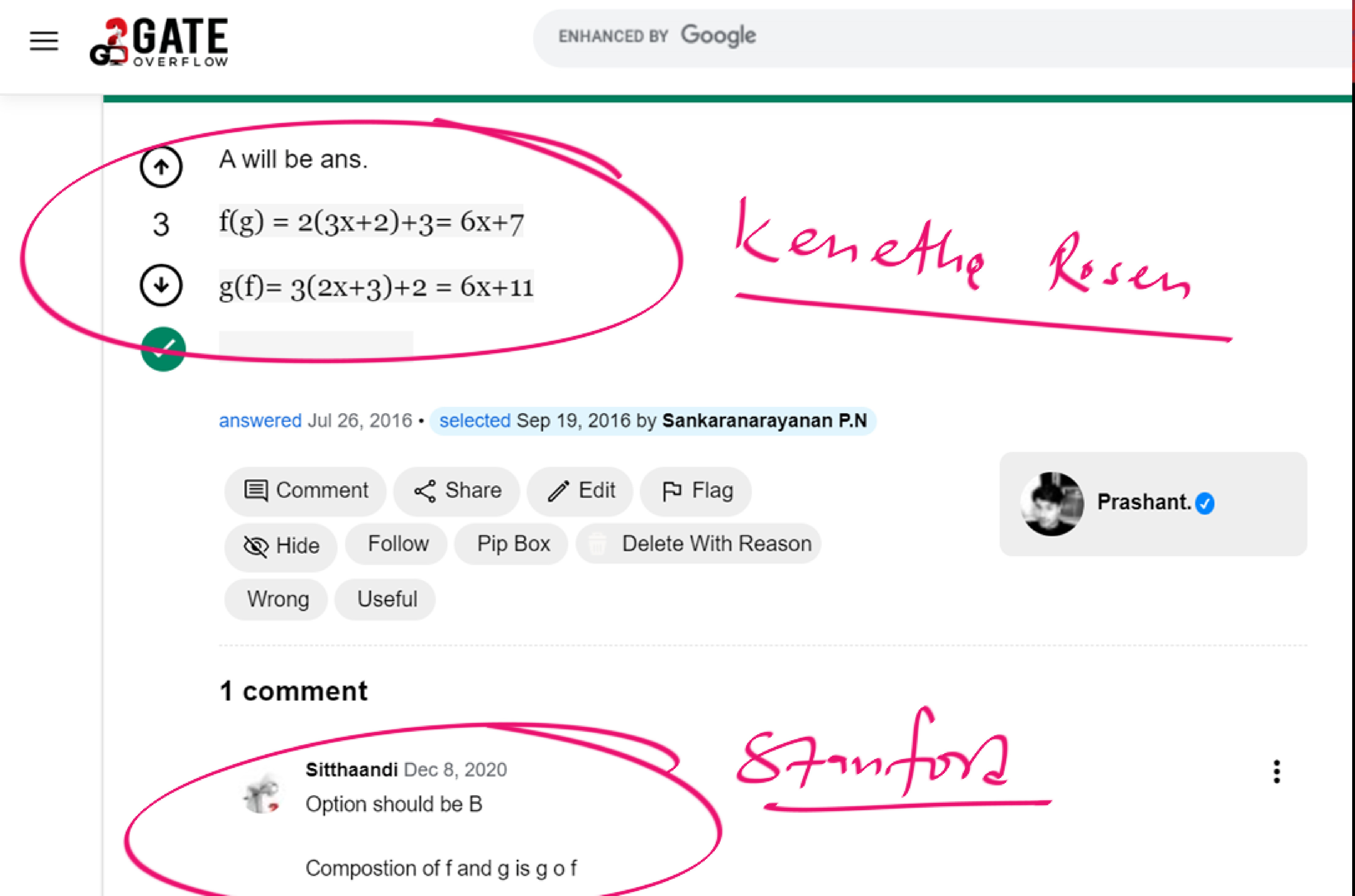
10.5.1 Function Composition: [redacted]

GATE future

Let f and g be the functions from the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. Then the composition of f and g and g and f is given as

A. 6x+7, 6x+11 ✓
B. 6x+11, 6x+7 ✓
C. 5x+5, 5x+5
D. None of the above

ugcnetcse-dec2013-paper2 algebra function-composition



3

A will be ans.

$f(g) = 2(3x+2)+3= 6x+7$

$g(f)= 3(2x+3)+2 = 6x+11$

answered Jul 26, 2016 • selected Sep 19, 2016 by **Sankaranarayanan P.N**

Kenethe Rosen

1 comment

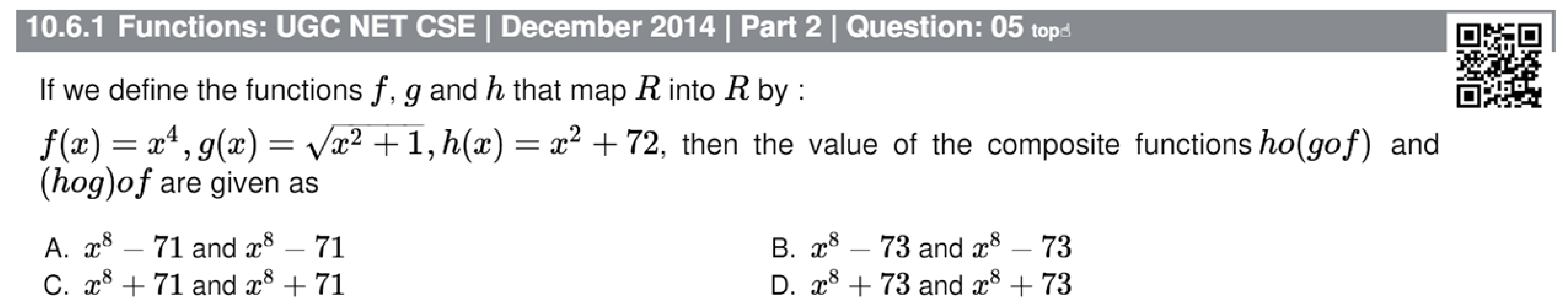
Sitthaandi Dec 8, 2020

Option should be B

Compostion of f and g is g o f

Stanford

GO Classes



10.6.1 Functions: UGC NET CSE | December 2014 | Part 2 | Question: 05 top

If we define the functions f, g and h that map R into R by :

$f(x) = x^4, g(x) = \sqrt{x^2 + 1}, h(x) = x^2 + 72$, then the value of the composite functions $ho(gof)$ and $(hog)of$ are given as

A. $x^8 - 71$ and $x^8 - 71$

B. $x^8 - 73$ and $x^8 - 73$

C. $x^8 + 71$ and $x^8 + 71$

D. $x^8 + 73$ and $x^8 + 73$

$$gof(x) = g(f(x)) = g(x^4) = \sqrt{x^8+1}$$

10.6.1 Functions: UGC NET CSE | December 2014 | Part 2 | Question: 05 top

If we define the functions f, g and h that map R into R by :

$f(x) = x^4, g(x) = \sqrt{x^2+1}, h(x) = x^2 + 72$, then the value of the composite functions $ho(gof)$ and $(hog)of$ are given as

A. $x^8 - 71$ and $x^8 - 71$

B. $x^8 - 73$ and $x^8 - 73$

C. $x^8 + 71$ and $x^8 + 71$

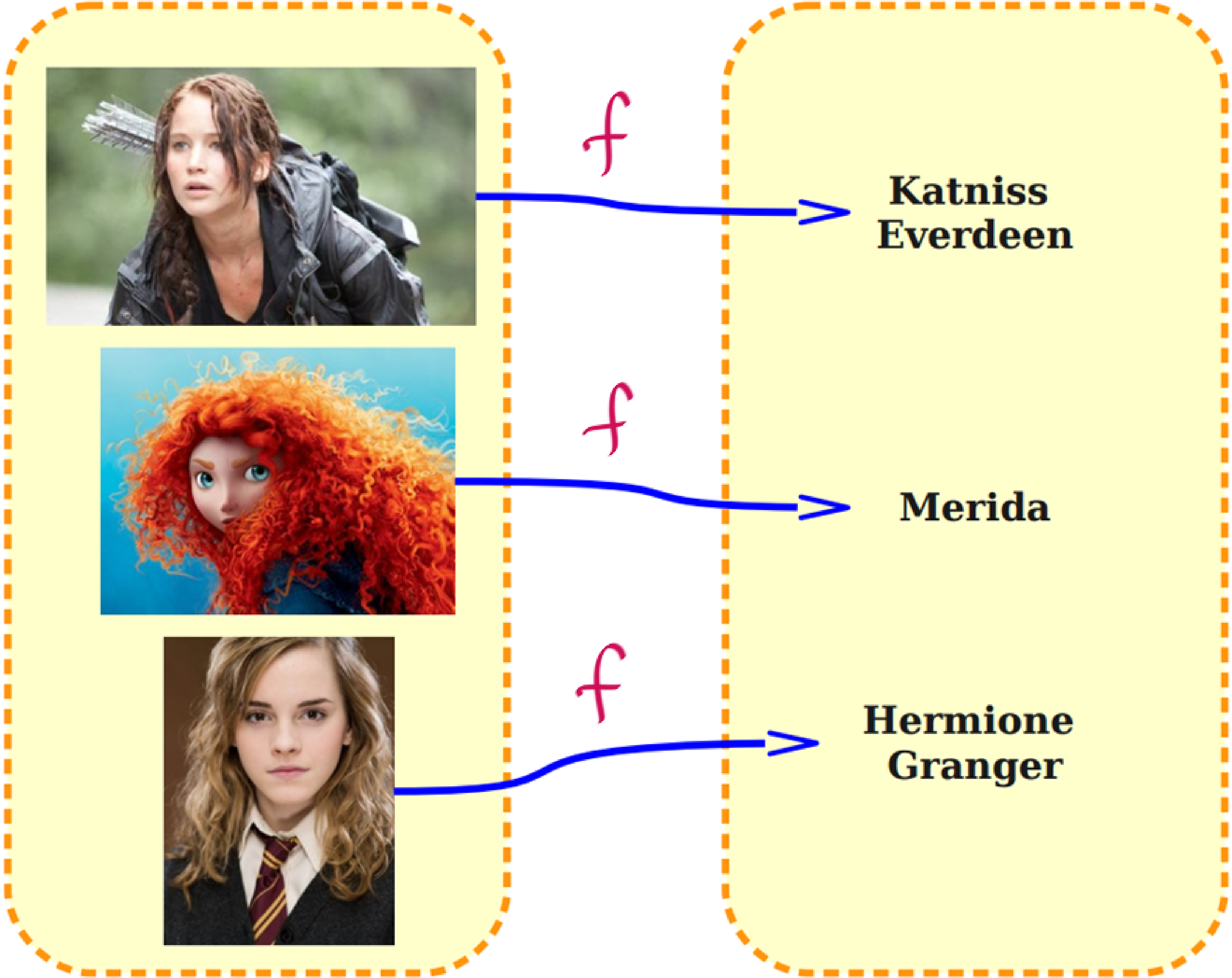
D. $x^8 + 73$ and $x^8 + 73$ ✓

$$ho(gof) = h(gof(x)) = h(\sqrt{x^8+1}) = x^8+1+72$$

$$\underbrace{h \circ (g \circ f)} = \underbrace{(h \circ g) \circ f}$$

Composition is <u>Associative</u>. ✓

Function Inverse



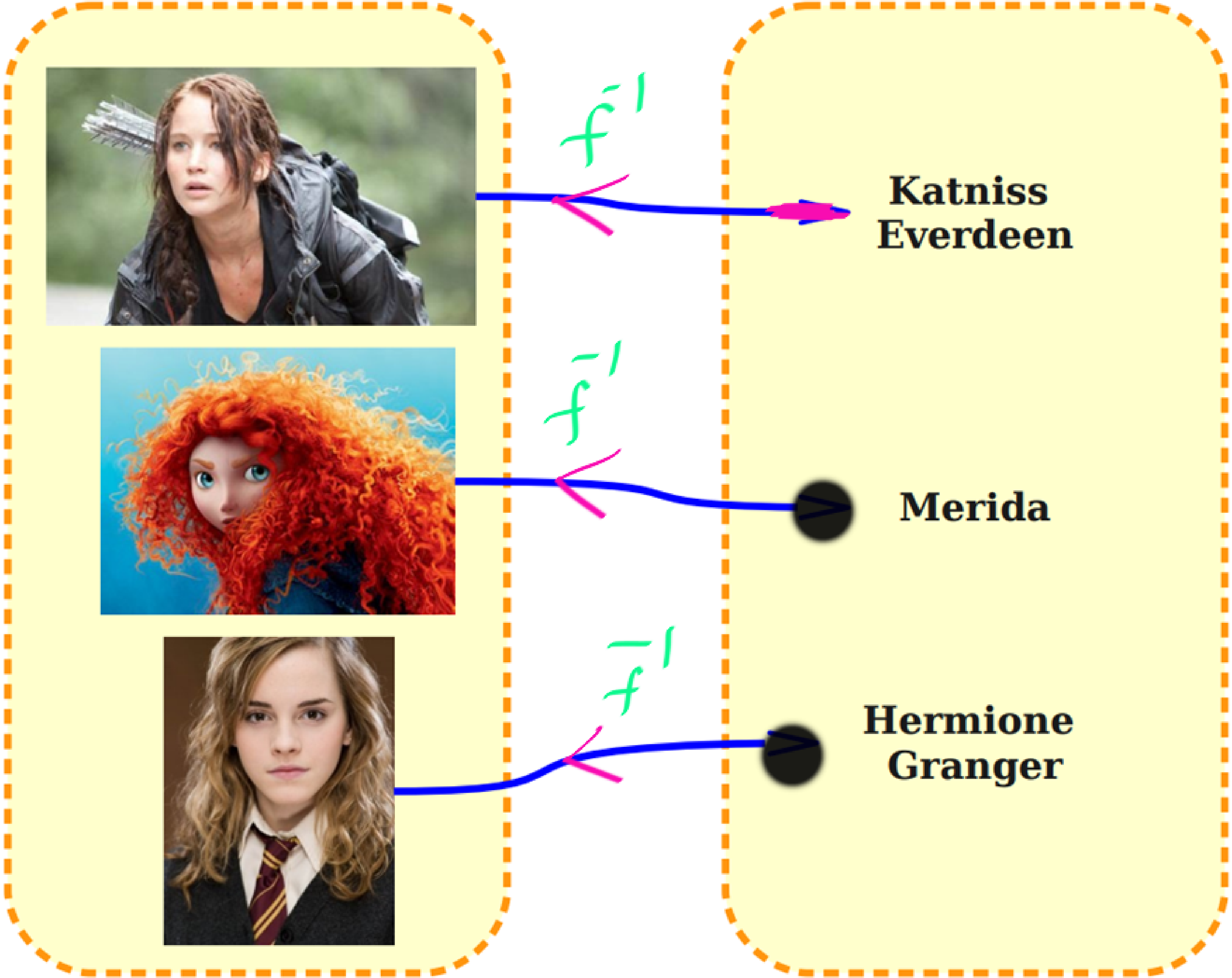
Katniss Everdeen

Merida

Hermione Granger

bijection

$f: \text{pic} \rightarrow \text{names}$



Katniss Everdeen

Merida

Hermione Granger

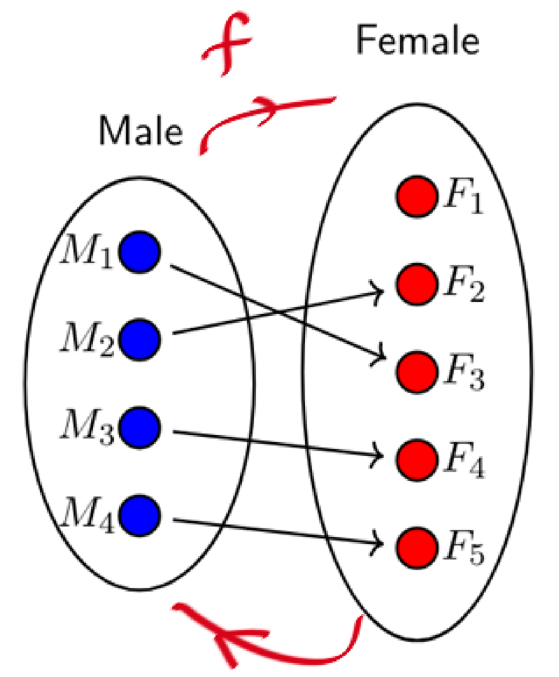
f^{-1} : names → pics

To Define f^{-1}, f must be bijective.

f^{-1} exists $\Longleftrightarrow$ f Invertible

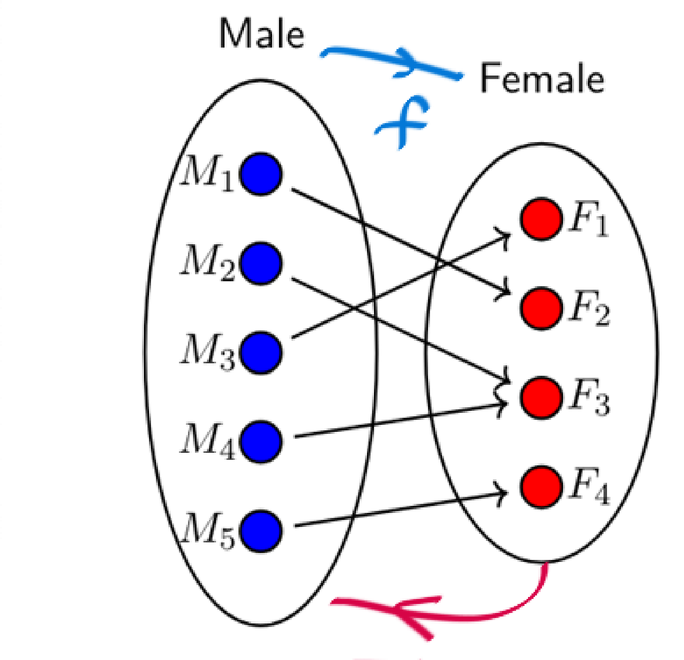
$\Longleftrightarrow$ f bijective.

Not Bijective



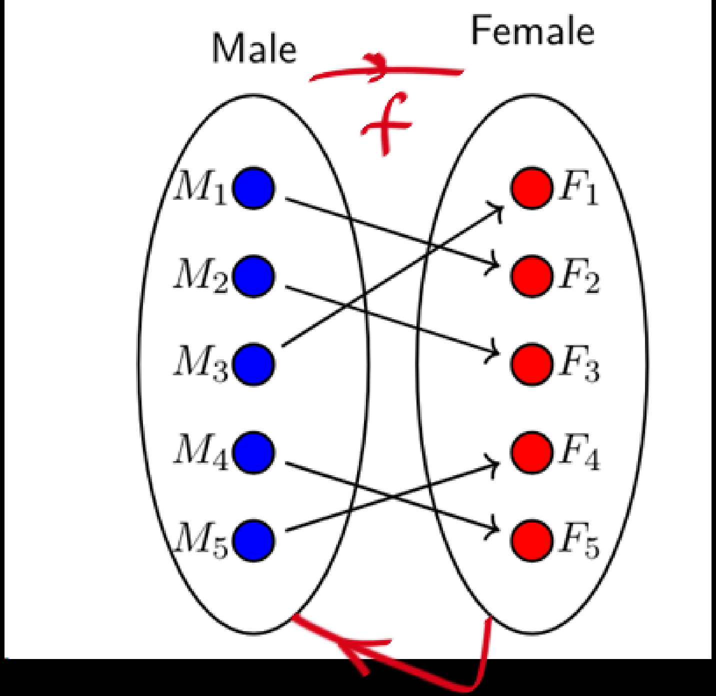
f^{-1}: NOT even a function

NOT bijection



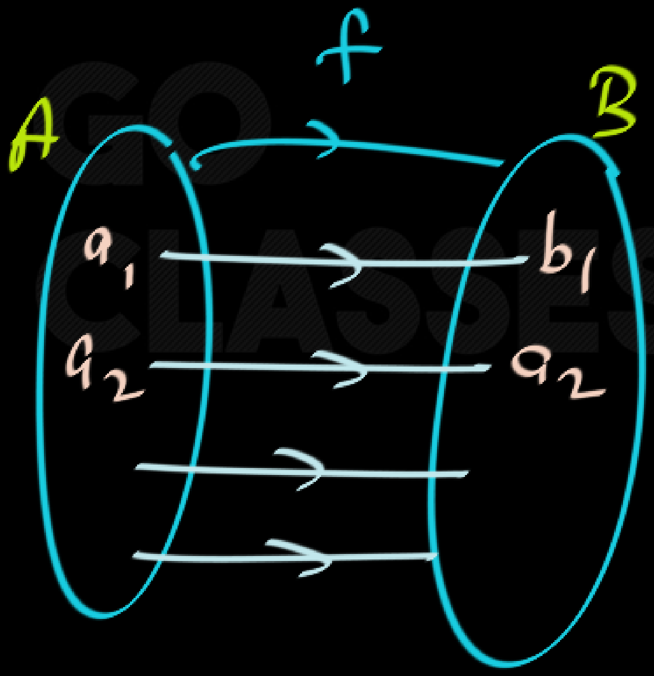
f^{-1}: NOT even a function

bijection

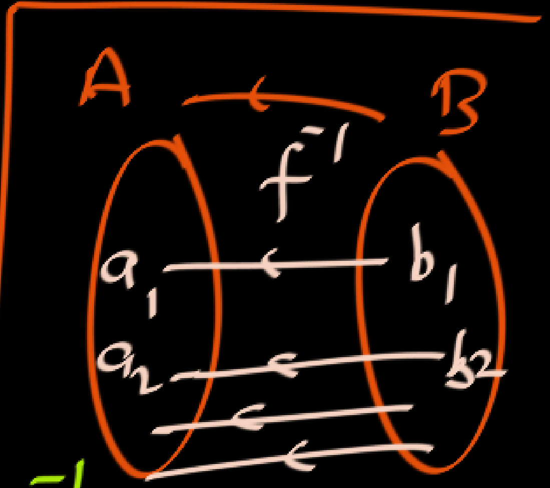


f^{-1}: a function

To find $\underbrace{f^{-1}}_{\text{Defined only for bijective functions}}$: Just **Reverse** mapping done by f.

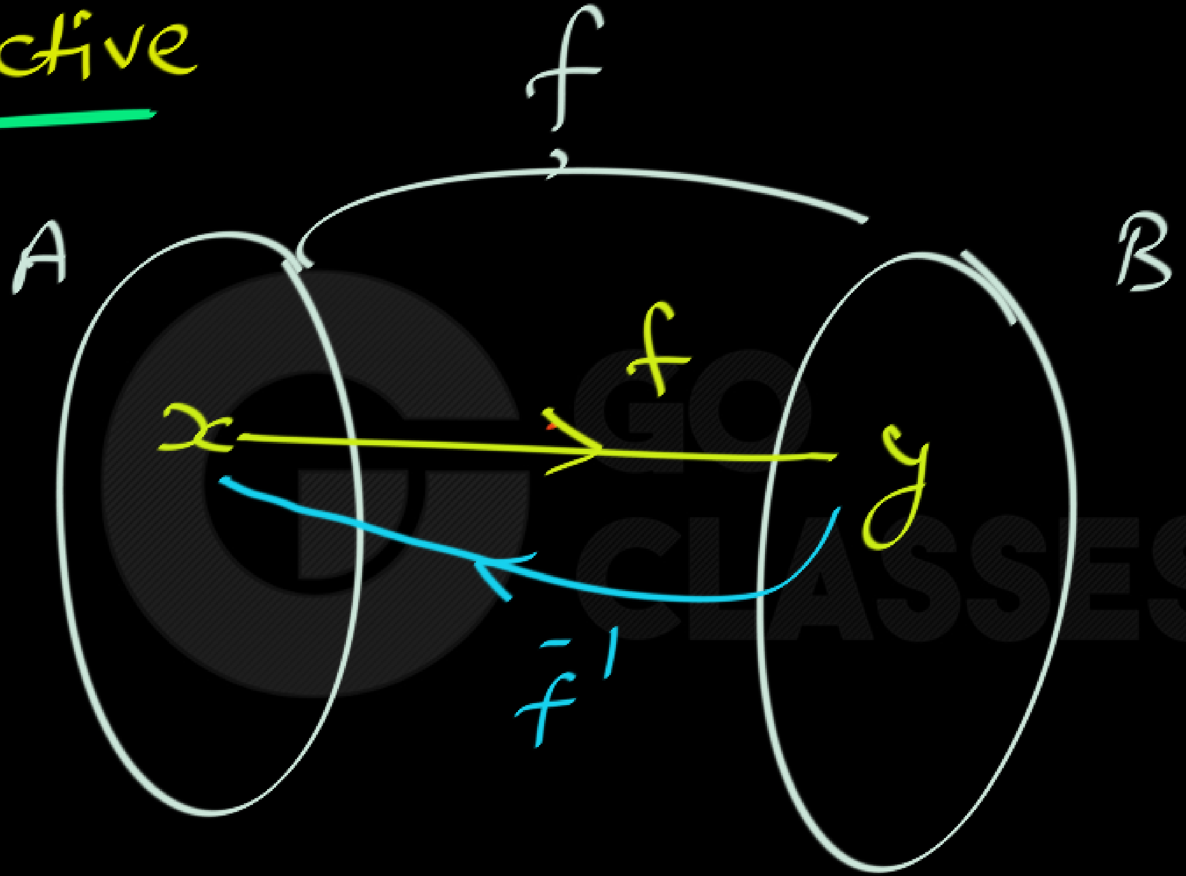


$f : A \rightarrow B$



$f^{-1} : B \rightarrow A$

f : bijective



$\boxed{f^{-1}}$: Just a Notation for $\underbrace{\text{inverse of } f}$.

→ Reverse mapping of f

$$f^{-1} \neq \frac{1}{f}$$

$$\boxed{f^{-1} \neq f \text{ power } -1}$$

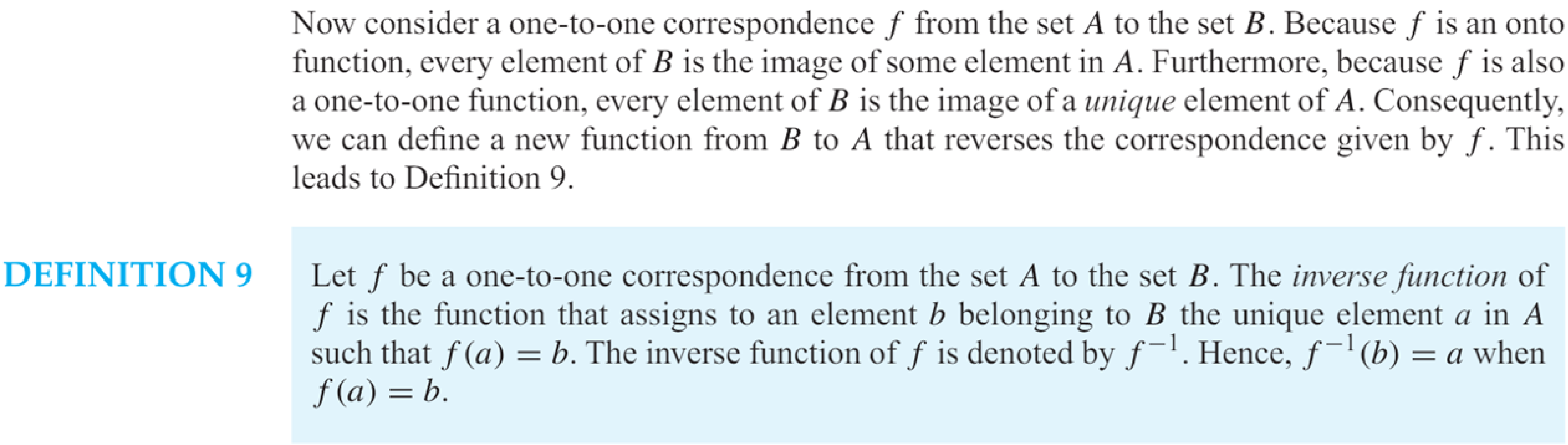
Note:

Function f is Invertible iff f is bijective.

Invertible function == Bijective function

Note:

If function f is a Invertible function, then in the Reverse Direction the function we get is called Inverse of f.

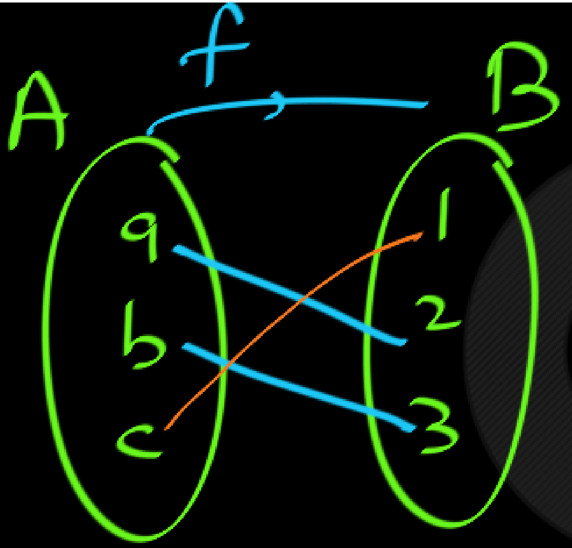


Now consider a one-to-one correspondence f from the set A to the set B. Because f is an onto function, every element of B is the image of some element in A. Furthermore, because f is also a one-to-one function, every element of B is the image of a *unique* element of A. Consequently, we can define a new function from B to A that reverses the correspondence given by f. This leads to Definition 9.

DEFINITION 9 Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

EXAMPLE 18 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

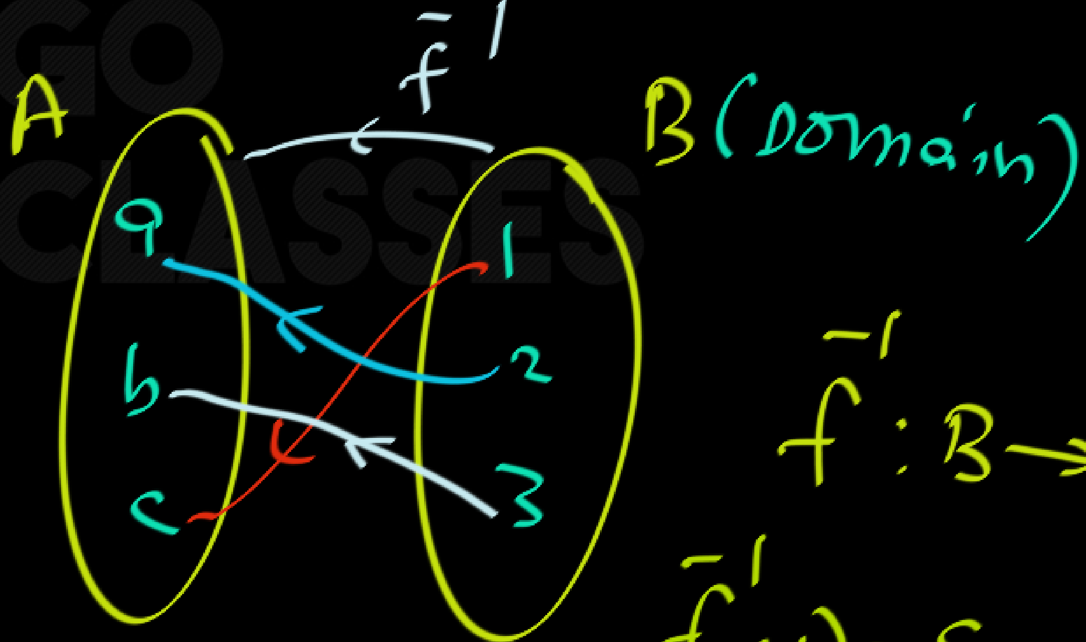
EXAMPLE 18 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?



bijective

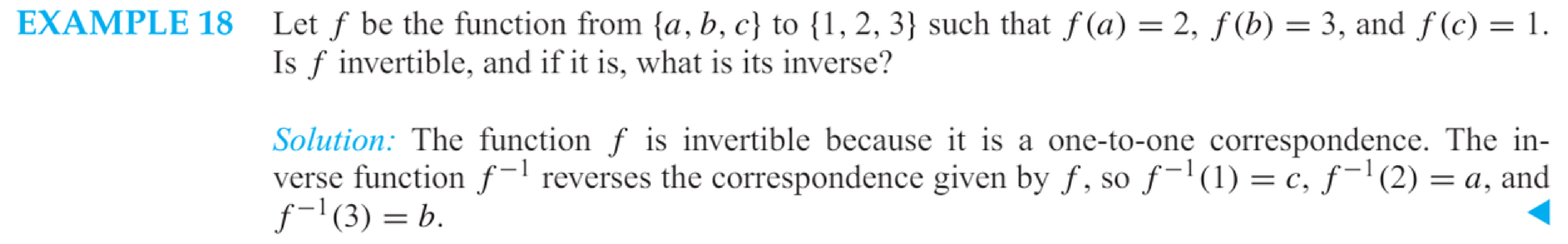
≡ Invertible

find Inverse : f^{-1}



$f^{-1} : B \to A$

$f^{-1}(1) = c$



EXAMPLE 18 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀

Note:

$$\underbrace{f(x) = x^2+1}$$

$$\underbrace{f(y) = y^2+1}$$

$$f(z) = z^2+1$$

$$\underbrace{f^{-1}(y) = y^2+y}$$

$$f^{-1}(x) = x^2+x$$

Finding inverse of a function:

Finding inverse of a function:

Given : Invertible function $f(x)$.

Question : find $f^{-1}(x)$

method : $f(x) = \text{-----} = y$

$$f(x) = y \Longrightarrow \boxed{x = f^{-1}(y)} \rightarrow \underbrace{f^{-1}(y) = \text{---}}_{f^{-1}(x)}$$

Given: $f(x)$

To find: $f^{-1}(x)$.

method: $f(x) = - - - - - = y$

$$f(x) = y \implies \boxed{f^{-1}(y) = x} \longrightarrow \underbrace{f^{-1}(y) = - - -}$$

$$f^{-1}(x) = - - -$$

EXAMPLE 19 Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

EXAMPLE 19 Let $\underline{f : \mathbf{Z} \to \mathbf{Z}}$ be such that $\underline{f(x) = x + 1}$. Is $\underline{f}$ invertible, and if it is, what is its inverse?

bijective

yes.

find $f^{-1}(x)$.

method:

$$f(x) = \underbrace{x + 1 = y}$$

$$x = y - 1$$

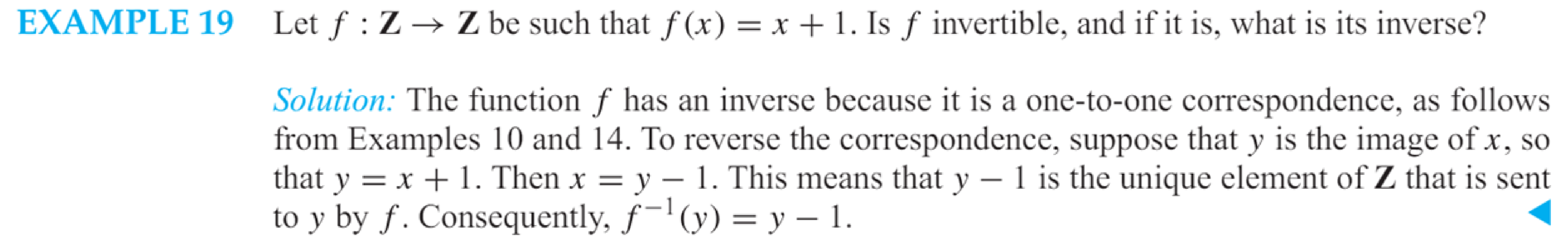
$$\underbrace{f(x) = y}$$

$$f^{-1}(y) = x = y - 1$$

$$f^{-1}(y) = y - 1$$

$$f^{-1}(x) = x - 1$$

Discrete Mathematics



EXAMPLE 19 Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 10 and 14. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$. ◀

EXAMPLE 20 Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible? No

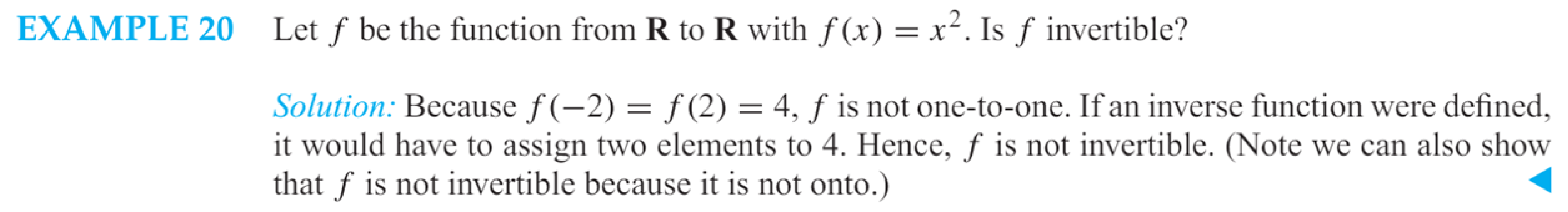
$f^{-1}(x)$ undefined

Not Injective

$f(2) = f(-2)$

Not onto

No preimage of -2.



EXAMPLE 20 Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.) ◀

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

bijection ≡ Invertible

$$f(x) = \underbrace{4x - 1 = y}$$

$$x = \frac{y+1}{4}$$

$$\underbrace{f^{-1}(y) = x}$$

$$\boxed{f^{-1}(y) = \frac{y+1}{4}} \rightarrow \boxed{f^{-1}(x) = \frac{x+1}{4}}$$

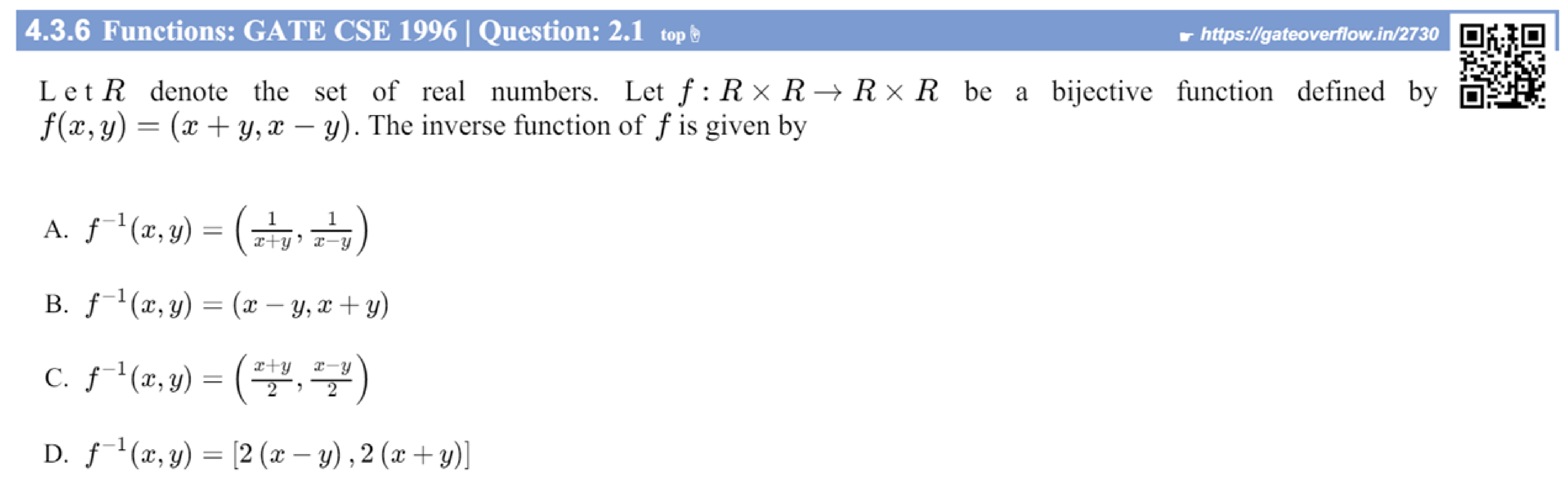
Problem

- Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 4x - 1$ for all $x \in \mathbb{R}$. Find its inverse function.

Proof

For any y in R, by definition of f^{-1}

- $f^{-1} =$ unique number x such that $f(x) = y$
  Consider $f(x) = y$
  $\implies 4x - 1 = y \qquad (\because \text{Defn. of } f)$
  $\implies x = \frac{y+1}{4} \qquad (\because \text{Simplify})$
- Hence, $f^{-1}(y) = \frac{y+1}{4}$ is the inverse function.



4.3.6 Functions: GATE CSE 1996 | Question: 2.1 top

https://gateoverflow.in/2730

Let R denote the set of real numbers. Let $f : R \times R \rightarrow R \times R$ be a bijective function defined by $f(x, y) = (x + y, x - y)$. The inverse function of f is given by

A. $f^{-1}(x, y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$

B. $f^{-1}(x, y) = (x - y, x + y)$

C. $f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

D. $f^{-1}(x, y) = [2\,(x - y)\,, 2\,(x + y)]$

Invertible ✓

4.3.6 Functions: GATE CSE 1996 | Question: 2.1 top

https://gateoverflow.in/2730

Let R denote the set of real numbers. Let $f : R \times R \rightarrow R \times R$ be a bijective function defined by $f(x, y) = (x + y, x - y)$. The inverse function of f is given by

Domain: R×R

CoDomain: R×R

A. $f^{-1}(x, y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$

B. $f^{-1}(x, y) = (x - y, x + y)$

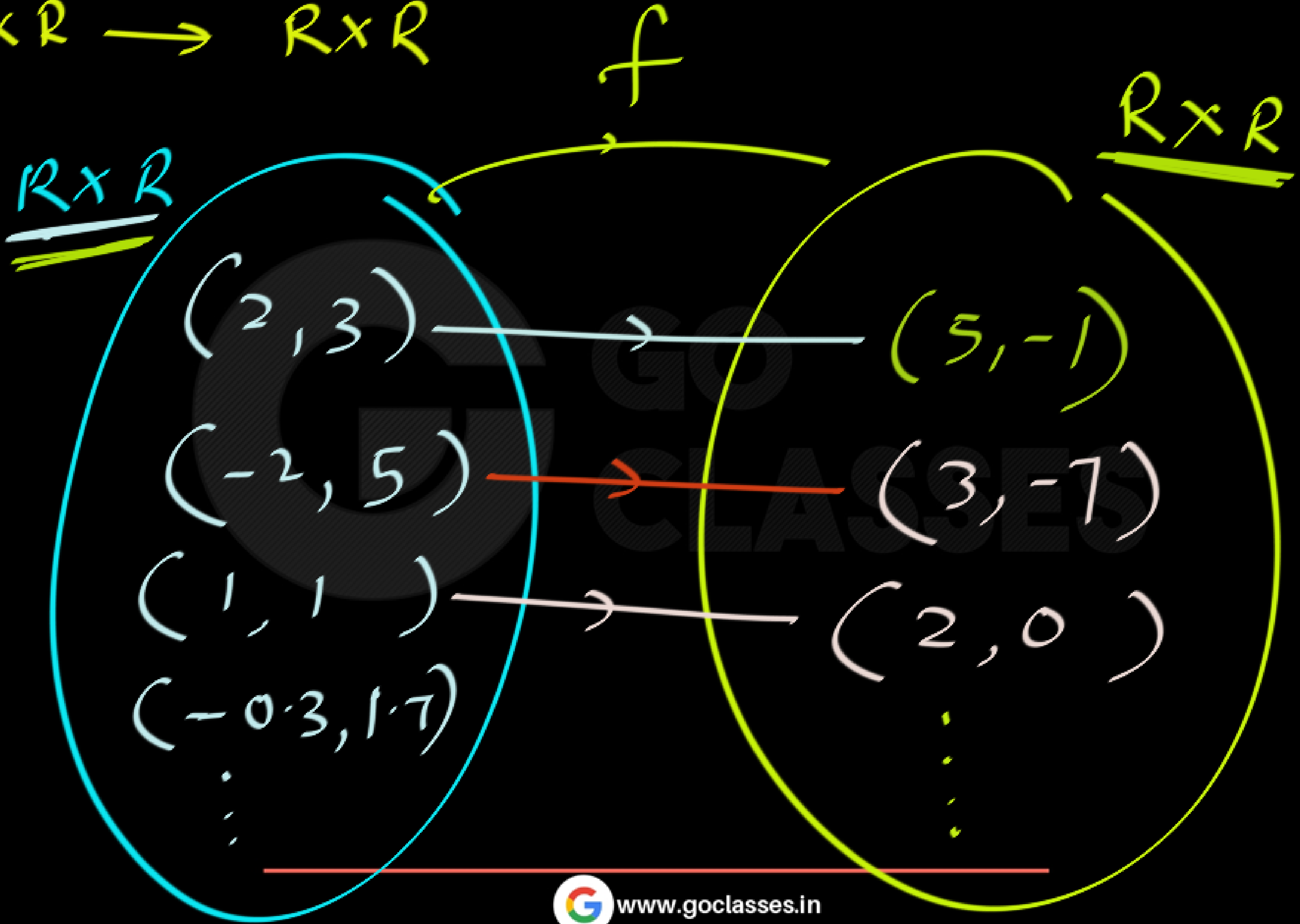
C. $f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ ✓

D. $f^{-1}(x, y) = [2(x - y), 2(x + y)]$

$f(x, y) = (x+y, x-y)$

$f(2,3) = (5, -1)$

$f : R \times R \rightarrow R \times R$



To find $f^{-1}(x)$:

$$f(x) = \text{- - -} = y$$

$$f^{-1}(y) = x \longrightarrow \underbrace{f^{-1}(y) = \text{- - -}}$$

$$\downarrow$$

$$f^{-1}(x) = \text{- - -}$$

$$f(x,y) = (x+y, x-y)$$

$$f(x,y) = (x+y, x-y) = (m,n)$$

$$x = \frac{m+n}{2} \; ; \; y = \frac{m-n}{2}$$

$$f(x,y) = (m,n)$$

$$f^{-1}(m,n) = (x,y) = \left(\frac{m+n}{2}, \frac{m-n}{2}\right)$$

$$(\underline{x+y},\ \underline{x-y}) = (\underline{m},\ \underline{n})$$

$$\left.\begin{aligned} m &= x+y \\ n &= x-y \end{aligned}\right\} \begin{aligned} &\rightarrow 2x = m+n \Rightarrow x = \frac{m+n}{2} \\ &\rightarrow 2y = m-n \Rightarrow y = \frac{m-n}{2} \end{aligned}$$

$$f^{-1}(m,n) = \left(\frac{m+n}{2}, \frac{m-n}{2}\right)$$

$$f^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

6.7. The function $f(n) = 2n$ is a bijection from $\mathbb{Z}$ to the even integers and the function $g(n) = 2n + 1$ is a bijection from $\mathbb{Z}$ to the odd integers. What are f^{-1}, g^{-1}

6.7. The function $f(n) = 2n$ is a bijection from $\mathbb{Z}$ to the even integers and the function $g(n) = 2n + 1$ is a bijection from $\mathbb{Z}$ to the odd integers. What are f^{-1}, g^{-1}

$f(n) = 2n$; find $f^{-1}(n) = ?$

$f(n) = 2n = y \rightarrow n = \frac{y}{2}$

$f^{-1}(y) = n \rightarrow \boxed{f^{-1}(y) = \frac{y}{2}} \rightarrow f^{-1}(n) = \frac{n}{2}$

$g(n) = 2n+1$; find $g^{-1}(n)$?

$$g(n) = \boxed{2n+1 = y} \longrightarrow n = \frac{y-1}{2}$$

$$\boxed{g(n) = y} \longrightarrow g^{-1}(y) = n = \frac{y-1}{2}$$

$$g^{-1}(y) = \frac{y-1}{2} \Longrightarrow g^{-1}(n) = \frac{n-1}{2} \checkmark$$

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

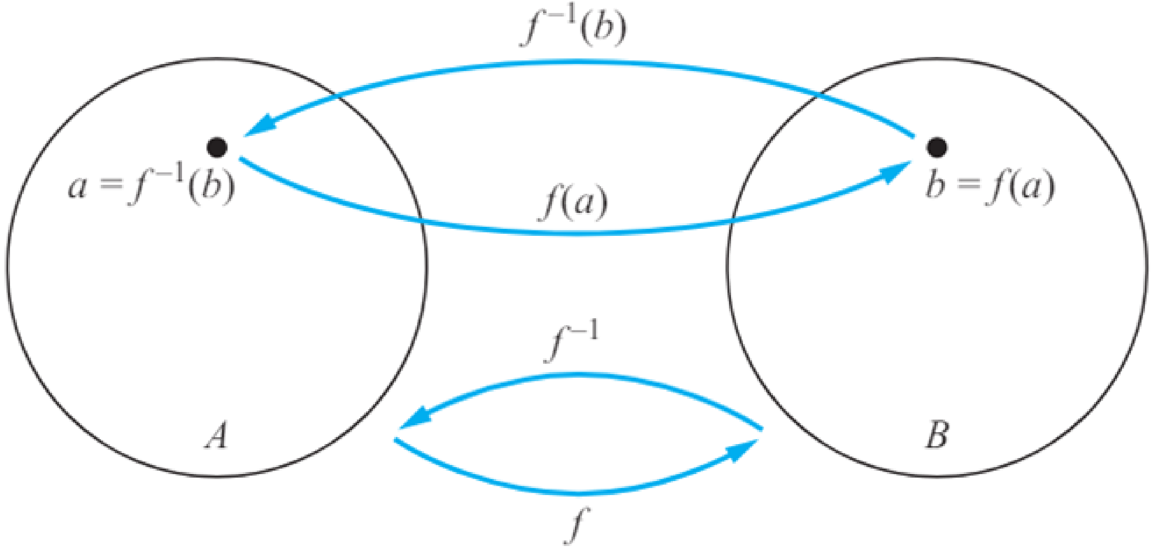
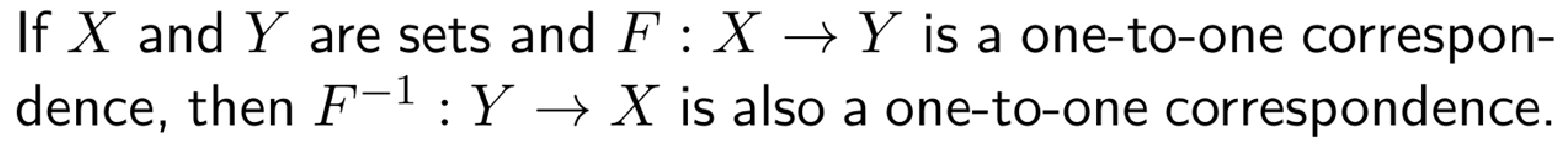


FIGURE 6 **The Function f^{-1} Is the Inverse of Function f.**

Question:

True or False??



If X and Y are sets and $F : X \to Y$ is a one-to-one correspondence, then $F^{-1} : Y \to X$ is also a one-to-one correspondence.

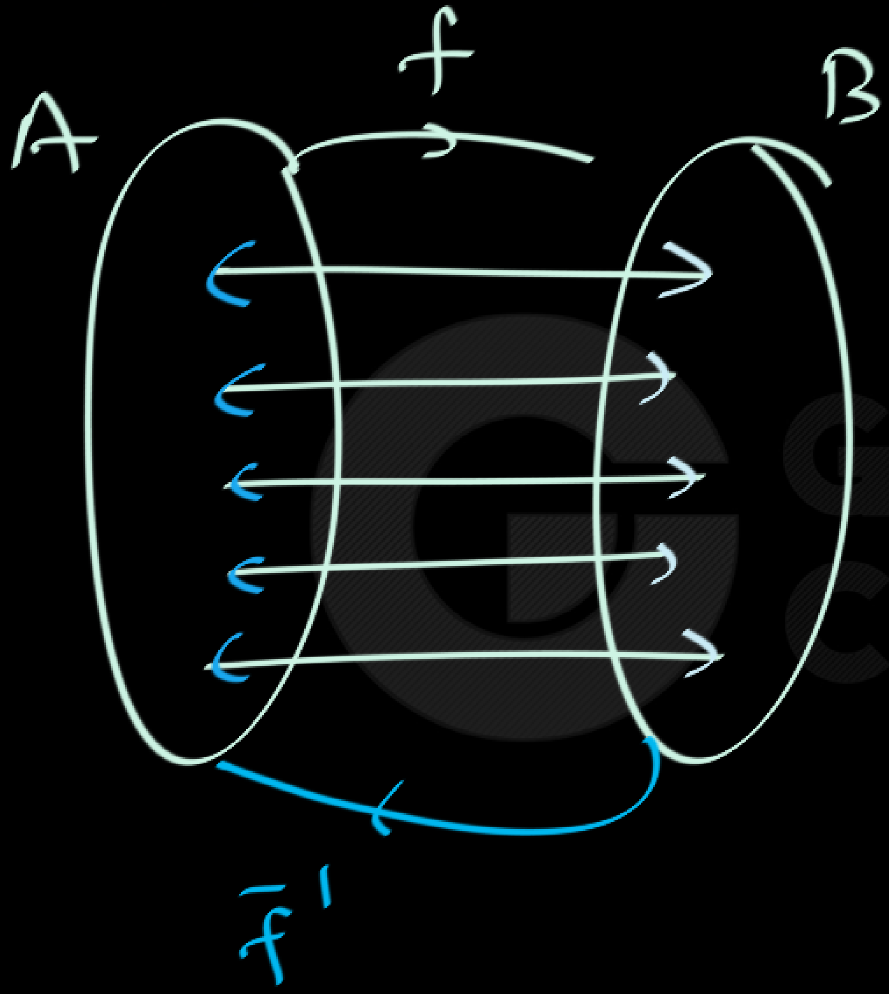
Question:

Invertible

True or False??

If X and Y are sets and $F : X \to Y$ is a one-to-one correspondence, then $F^{-1} : Y \to X$ is also a one-to-one correspondence.

exists



f : 1-1

then

f^{-1} : Reverse mapping is also 1-1.

Inverse functions

Theorem

- If X and Y are sets and $F : X \to Y$ is a one-to-one correspondence, then $F^{-1} : Y \to X$ is also a one-to-one correspondence.

Question:

If function f is invertible,

What is $(f^{-1})^{-1}$ = ?? $= f$

Theorem 7. *Let* A *and* B *be nonempty sets, and suppose* $f: A \to B$ *is invertible. Then* $f^{-1}: B \to A$ *is also invertible, and* $(f^{-1})^{-1} = f$.

A function f is one-to-one and onto if f has an inverse.

True False

If a function g is mapped from set A to set B, and g is not a bijection function, then $|A| \neq |B|$.

True False

A function f is one-to-one and onto if f has an inverse.

bijection

f : Invertible

[✓ True] False

If a function g is mapped from set A to set B, and g is not a bijection function, then $|A| \neq |B|$.

True [✓ False]