Set

Summary Lecture

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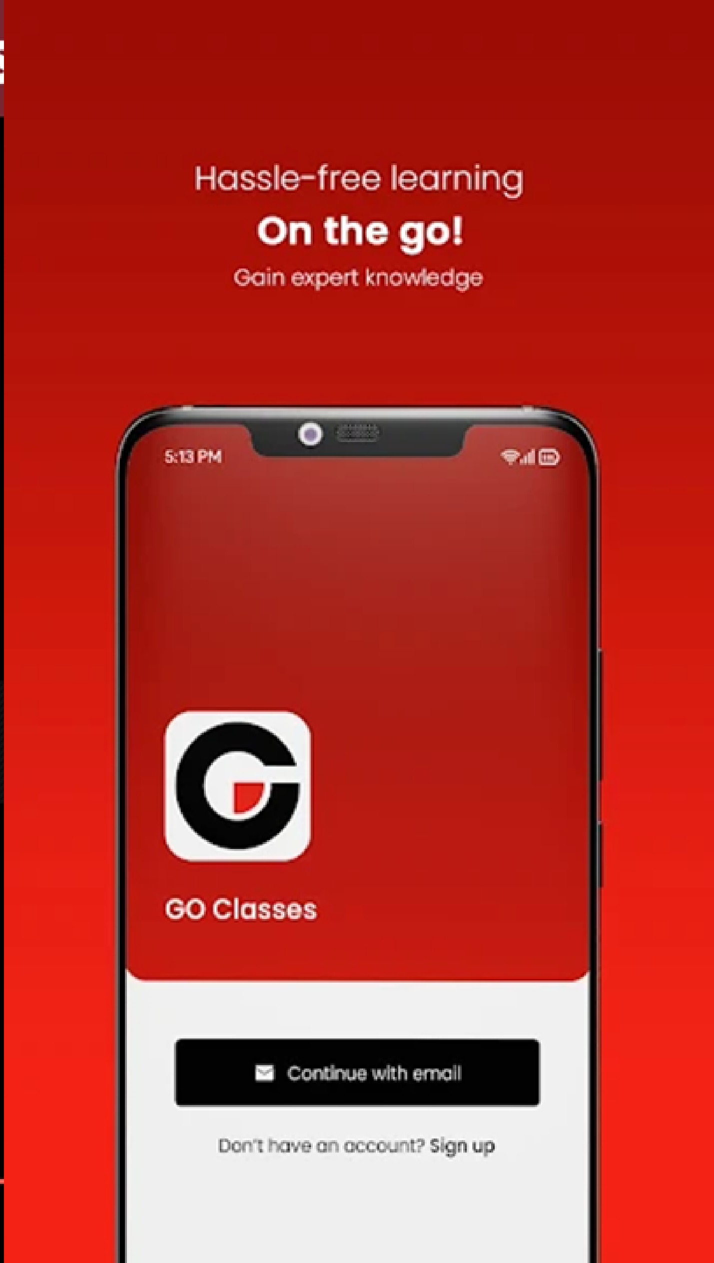

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Set Theory

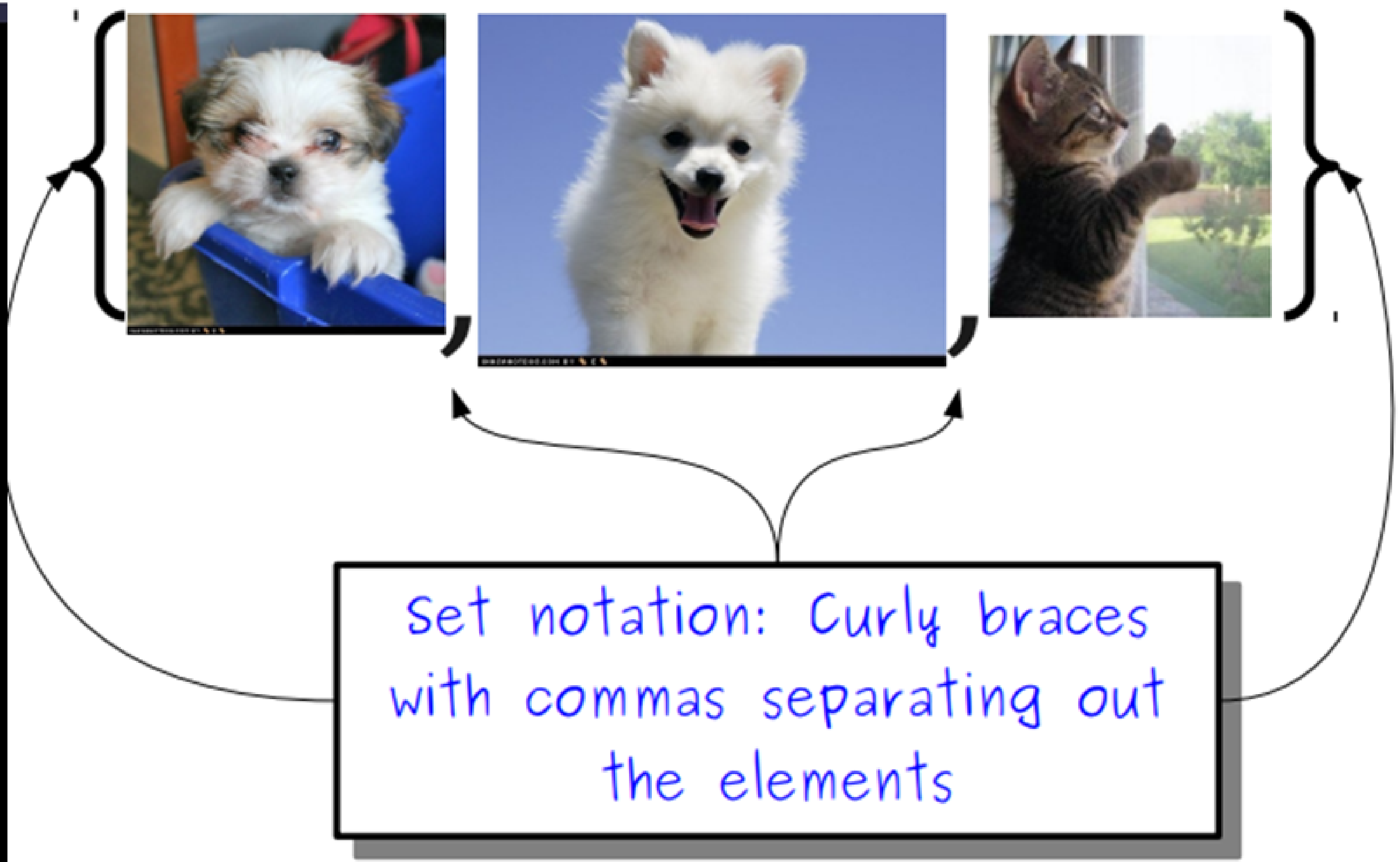
Summary

Set Basics

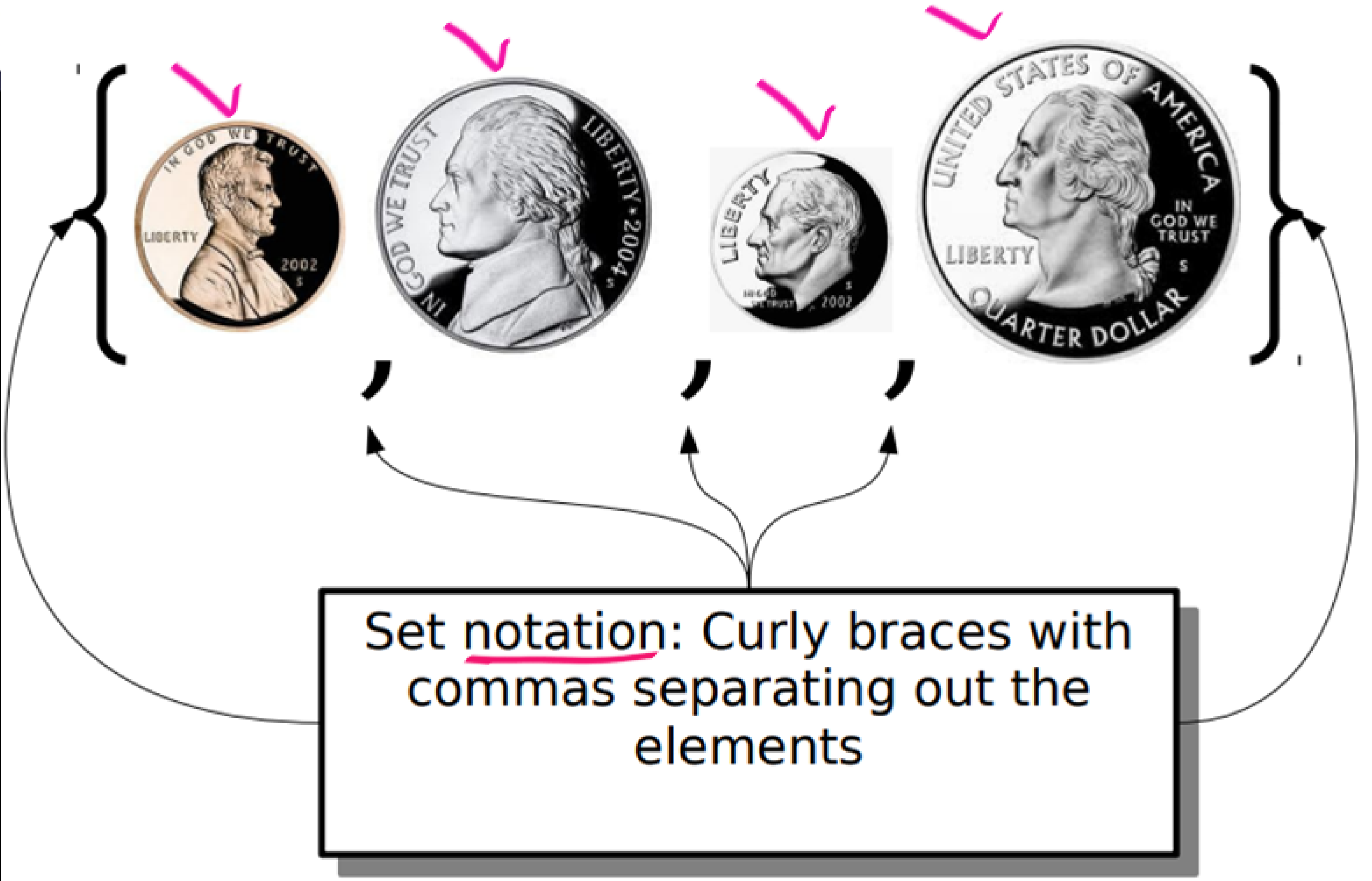
Set (Definition):

A set is an unordered collection of distinct objects, which may be anything.

Set notation: Curly braces with commas separating out the elements.



A **set** is an unordered collection of distinct objects, which may be anything (including other sets).



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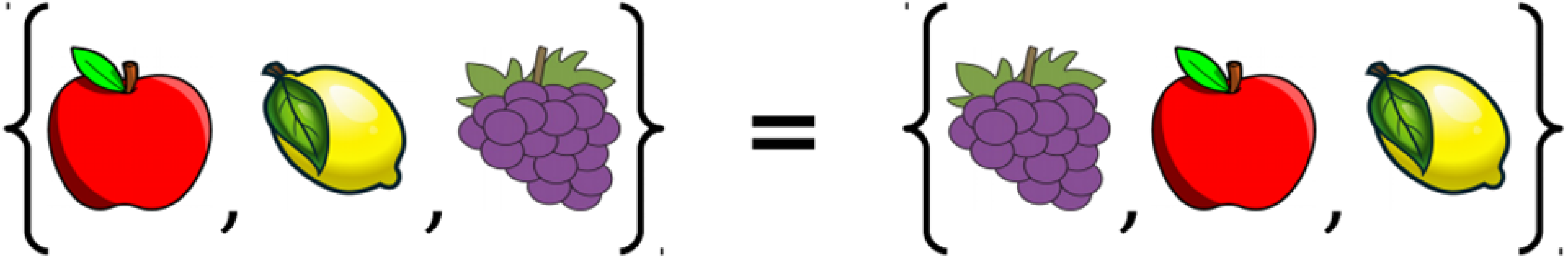
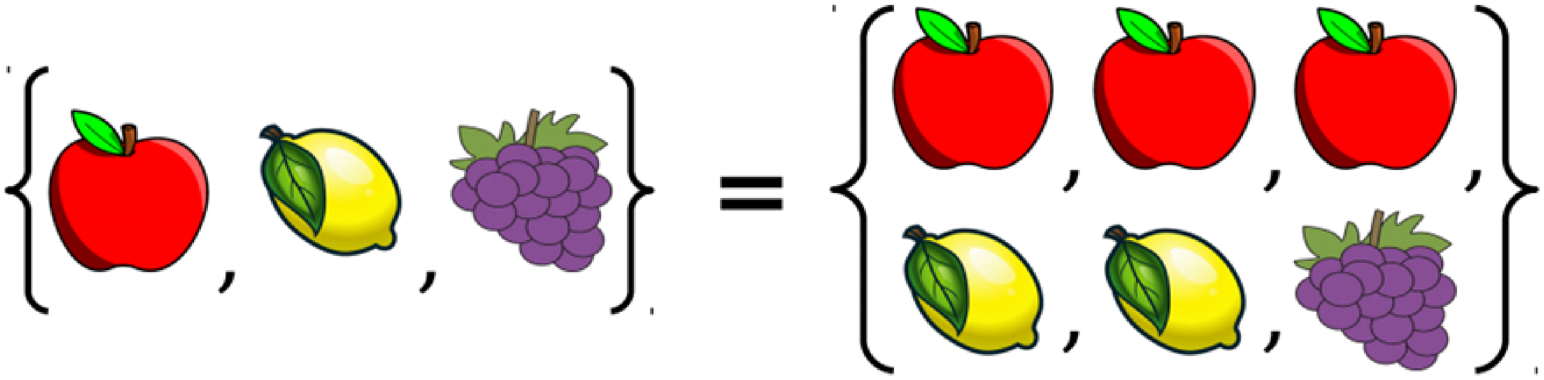


These are two different descriptions of exactly the same set.

Two sets are equal when they have the same contents, ignoring order.

A **set** is an **unordered** collection of distinct objects, which may be anything (including other sets).

A **set** is an *unordered* collection of distinct objects, which may be anything (including other sets).



These are also two different descriptions of exactly the same set.
(But please use the description without duplication :-))

Sets cannot contain duplicate elements.
Any repeated elements are ignored.

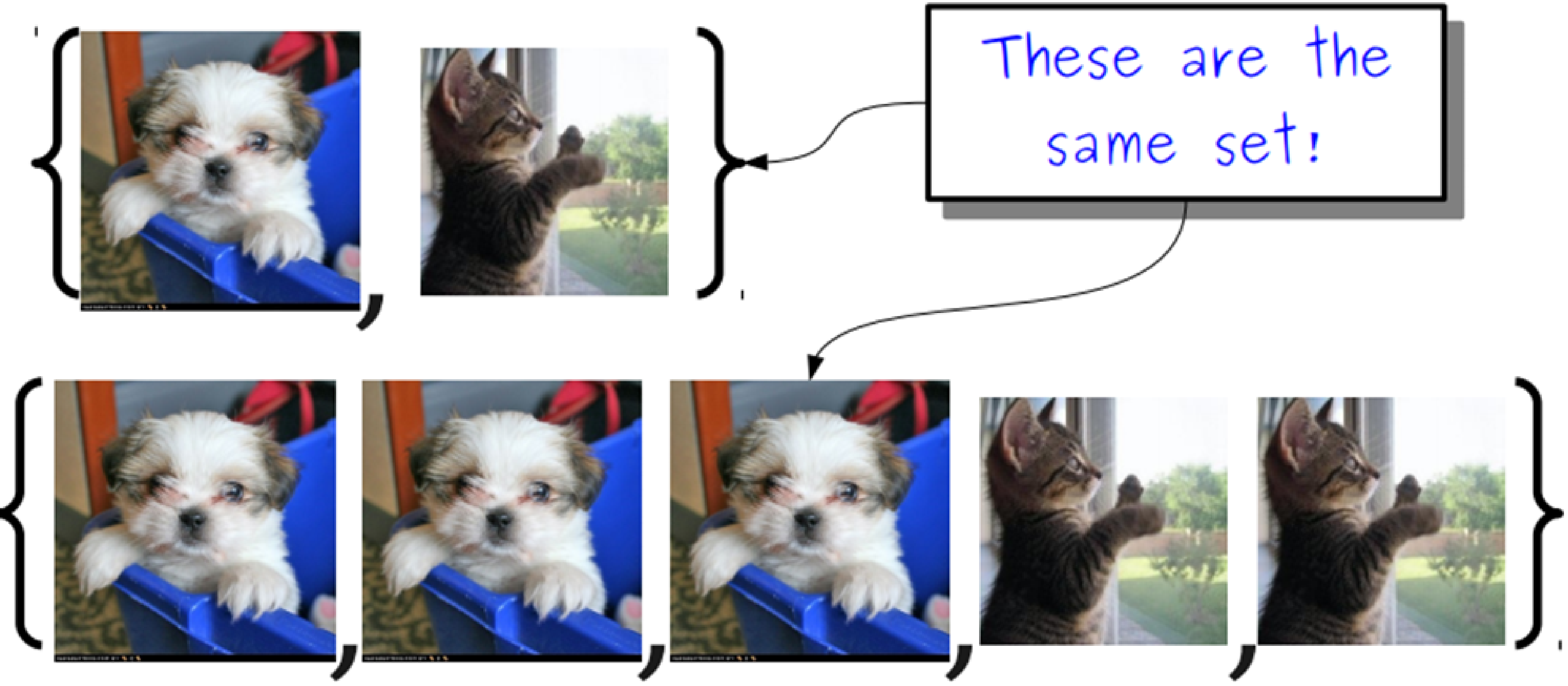
Set $\{1, 1, 2, 3, 4, 4\}$

→ How many elements? =

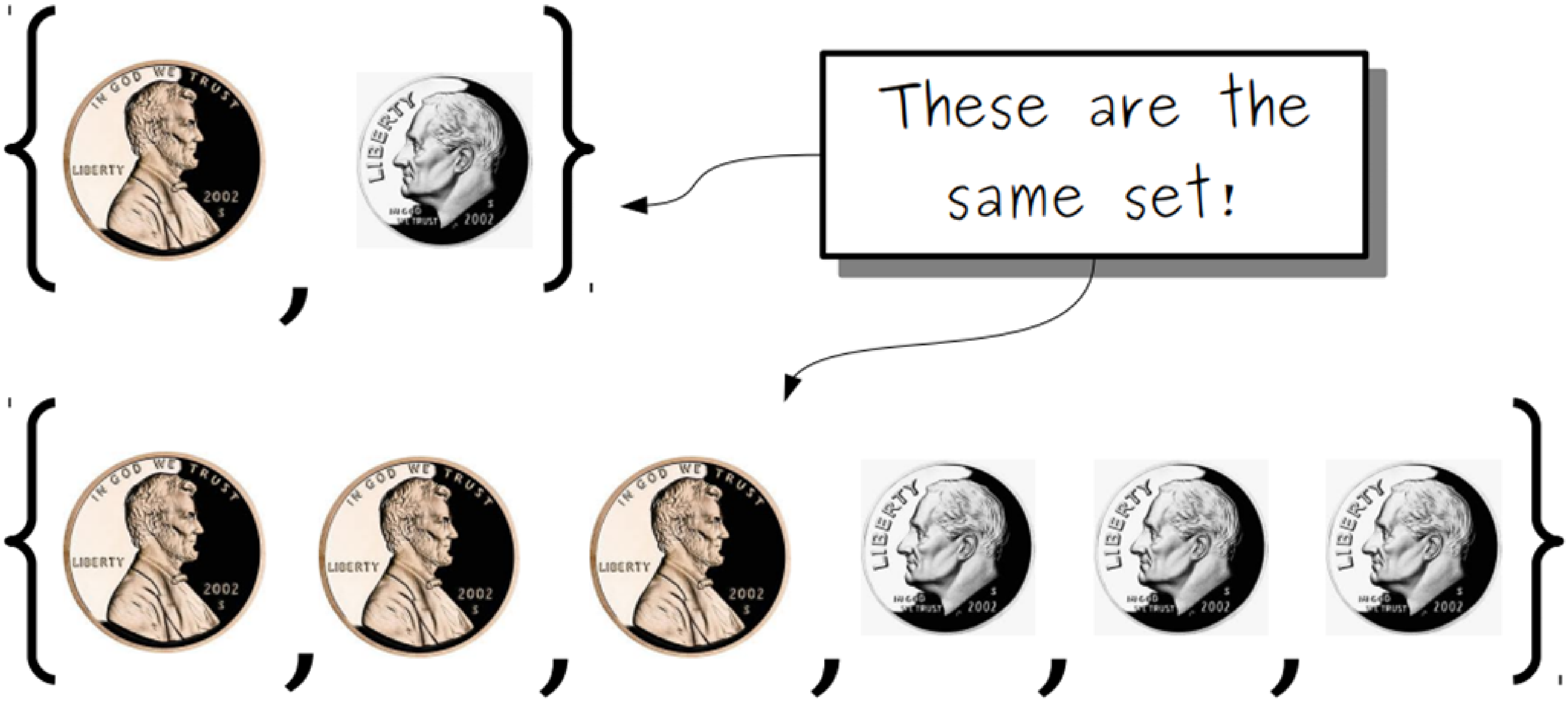
Set $\{1, 1, 2, 3, 4, 4\} = \{1, 2, 3, 4\}$

How many elements? = 4

Cardinality = 4



A **set** is an unordered collection of **distinct** objects, which may be anything (including other sets).



A **set** is an unordered collection of *distinct* objects, which may be anything (including other sets).

The objects that make up a set are called the ***elements*** of that set.

$S = \{1, 2, a, b\}$ = set / <u>Collection</u>

element of S

$1 \in S$

belongs to

$3 \notin S$

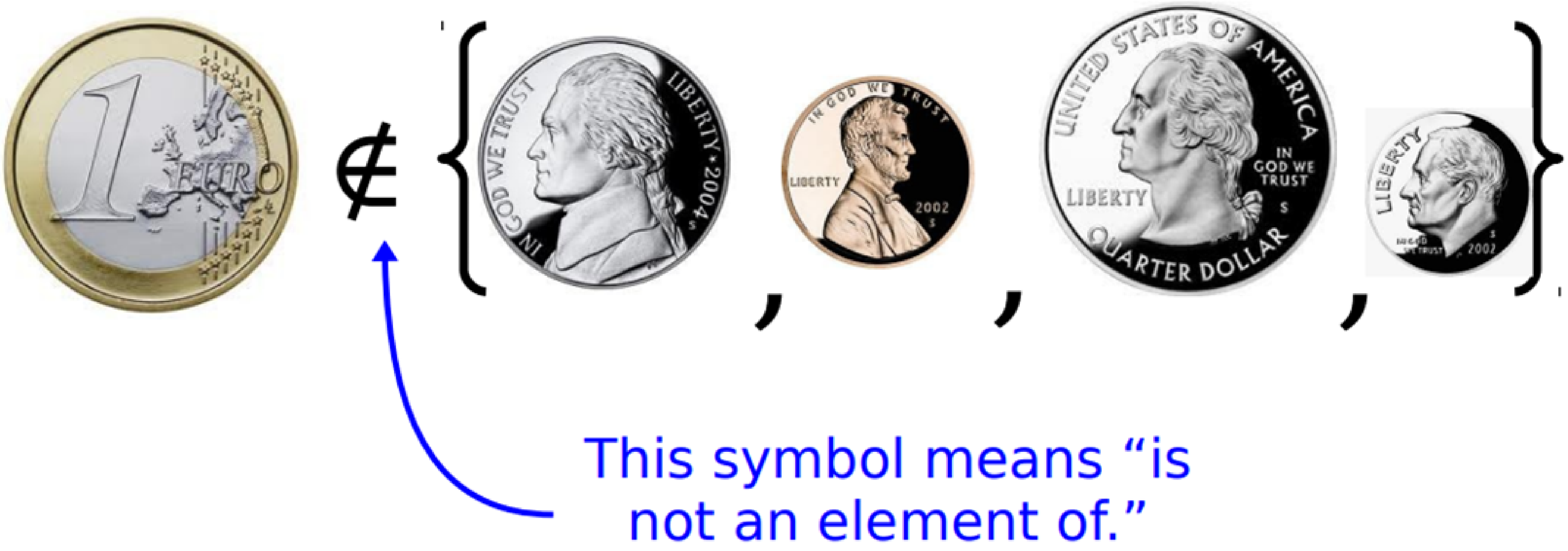
$c \notin S$

$b \in S$

John $\notin S$

This symbol means "is an element of."

The objects that make up a set are called the ***elements*** of that set.

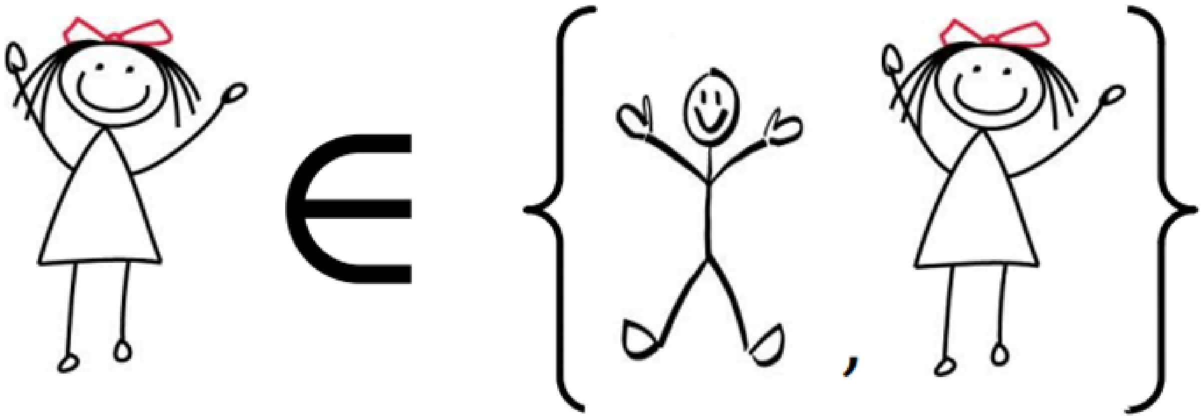


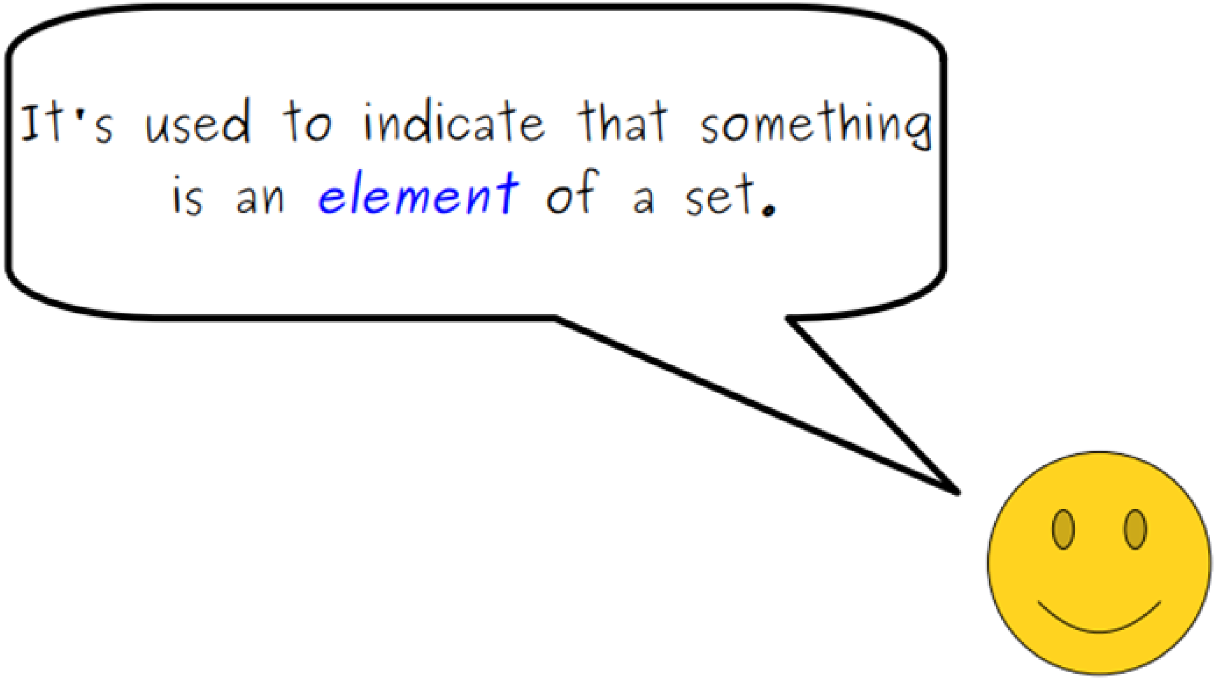
The objects that make up a set are called the ***elements*** of that set.

$$\in$$

This symbol is the "element-of" symbol.

GO Classes





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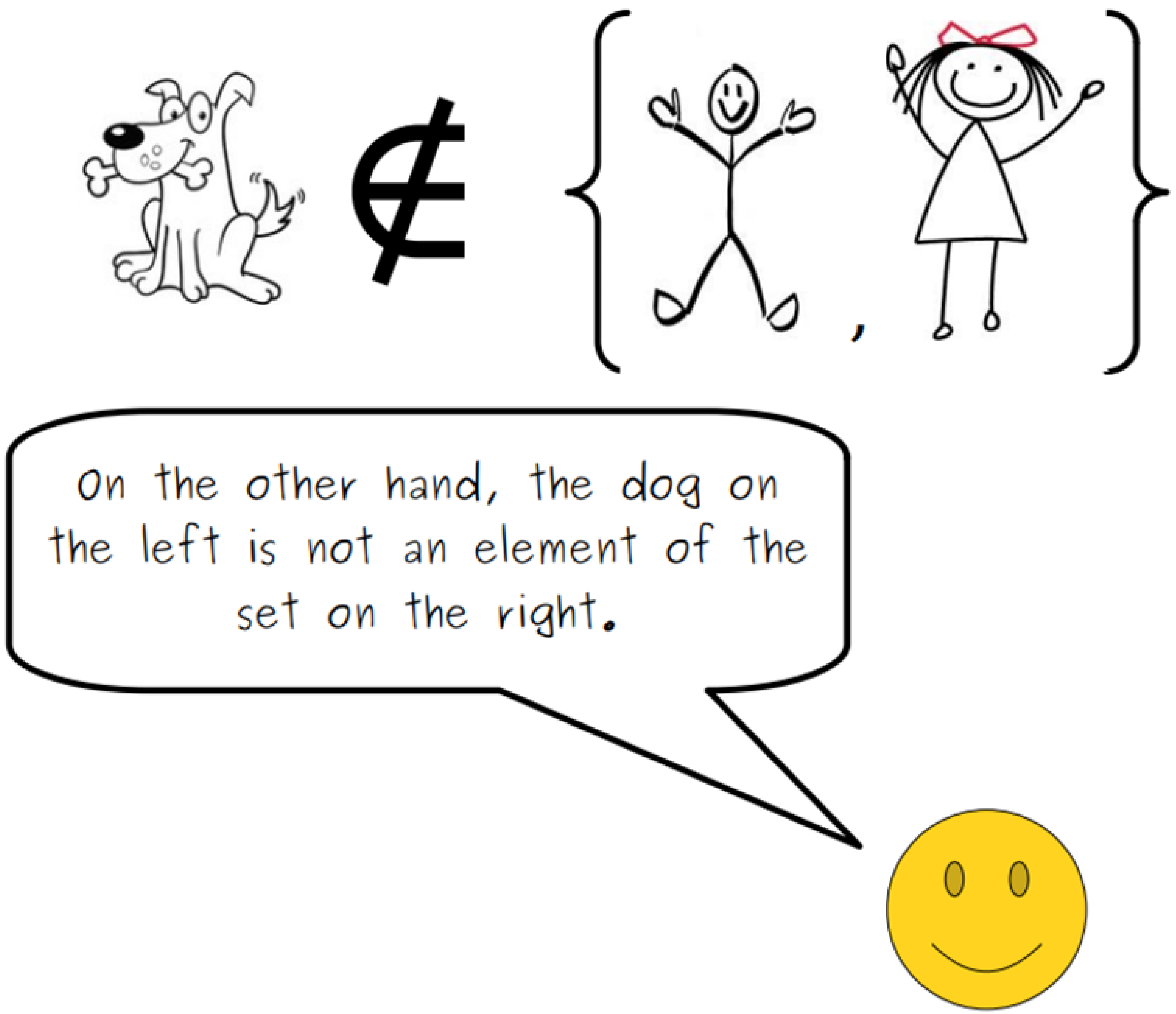

On the other hand, the dog on the left is not an element of the set on the right.

Definition 2.2 *The* **set membership symbol** $\in$ *is used to say that an object is a member of a set. It has a partner symbol* $\notin$ *which is used to say an object is not in a set.*

Set Membership

- Given a set S and an object x, we write

$$x \in S$$

if x is contained in S, and

$$x \notin S$$

otherwise.

- If $x \in S$, we say that x is an **element** of S.
- Given any object and any set, either that object is in the set or it isn't.

Note: Set $S = \{ \; , \; , \; , \ldots \}$

for any 'a':

$a \in S$ OR $a \notin S$

BUT NOT Both

Note: Set $S = \{ \; , \; , \; , \ldots \}$

for any 'a':

$a \in S$ and $a \notin S$

Paradox (Contradiction)

Finite Set

A set with a limited number of elements

Example: $A = \{Dog, Cat, Fish, Frog\}$

Infinite Set

A set with an unlimited number of elements

Example: $N = \{1, 2, 3, 4, 5, \ldots\}$

Cardinality :

Cardinality

finite

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality as $|S|$.
- Examples:
  - $|\{whimsy, mirth\}| = 2$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$

Definition 2.4 *The* **cardinality** *of a set is its* size. *For a finite set, the cardinality of a set is the number of members it contains. In symbolic notation the size of a set S is written $|S|$. We will deal with the idea of the cardinality of an infinite set later.*

$$1 \stackrel{?}{=} \{1\}$$

Question: Are these objects equal?

$$1 \stackrel{?}{=} \{1\}$$

a number

Not a set

Collection containing one element "1"

$1 \neq \boxed{1}$

Set ≡ Box

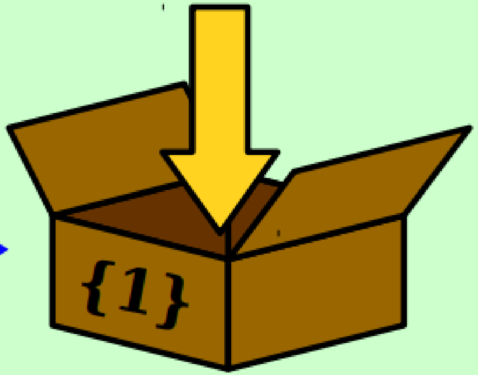
Question: Are these objects equal?

$$1 \neq \{1\}$$

This is a number.

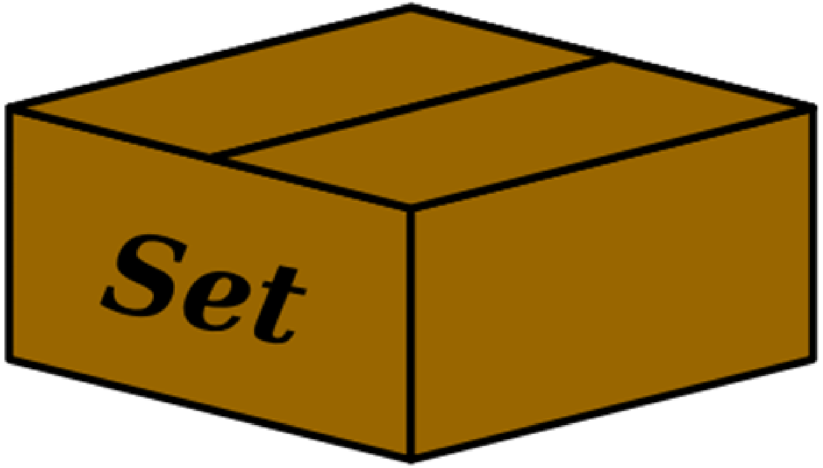
1

1



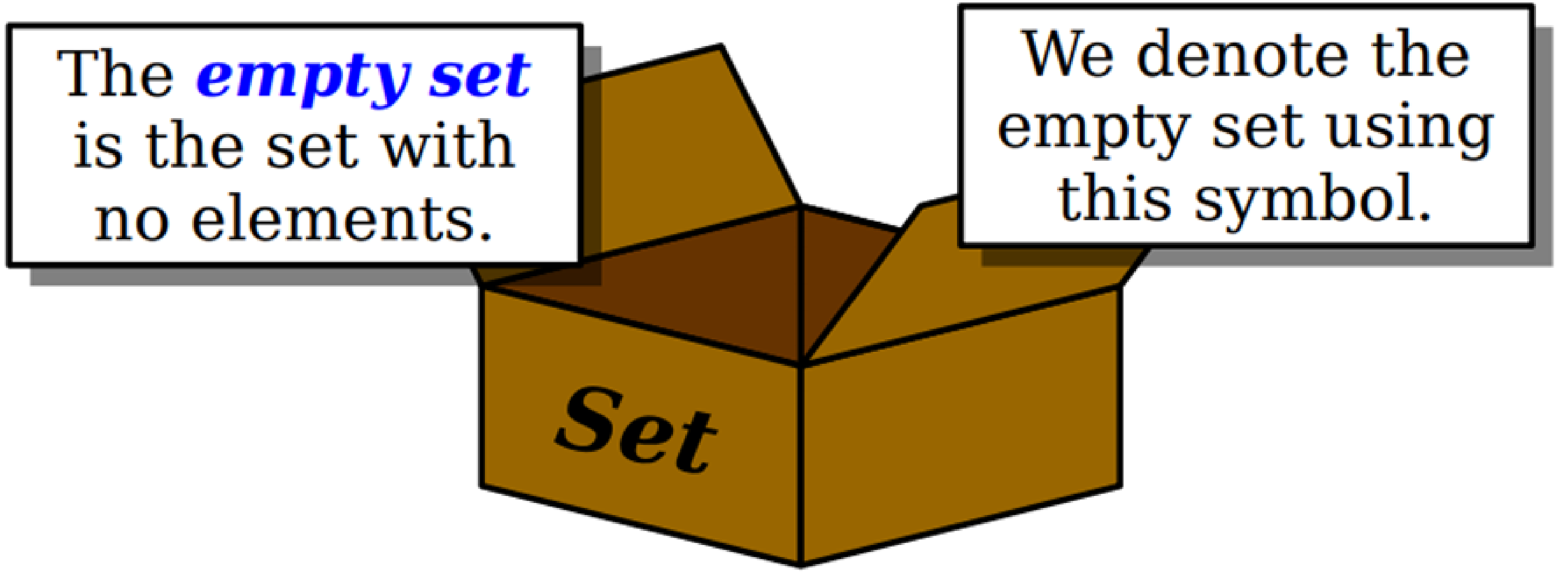
This is a set. It contains a number.

Question: Are these objects equal?



Sets can contain any number of elements.

$$\{\ \} = \emptyset$$



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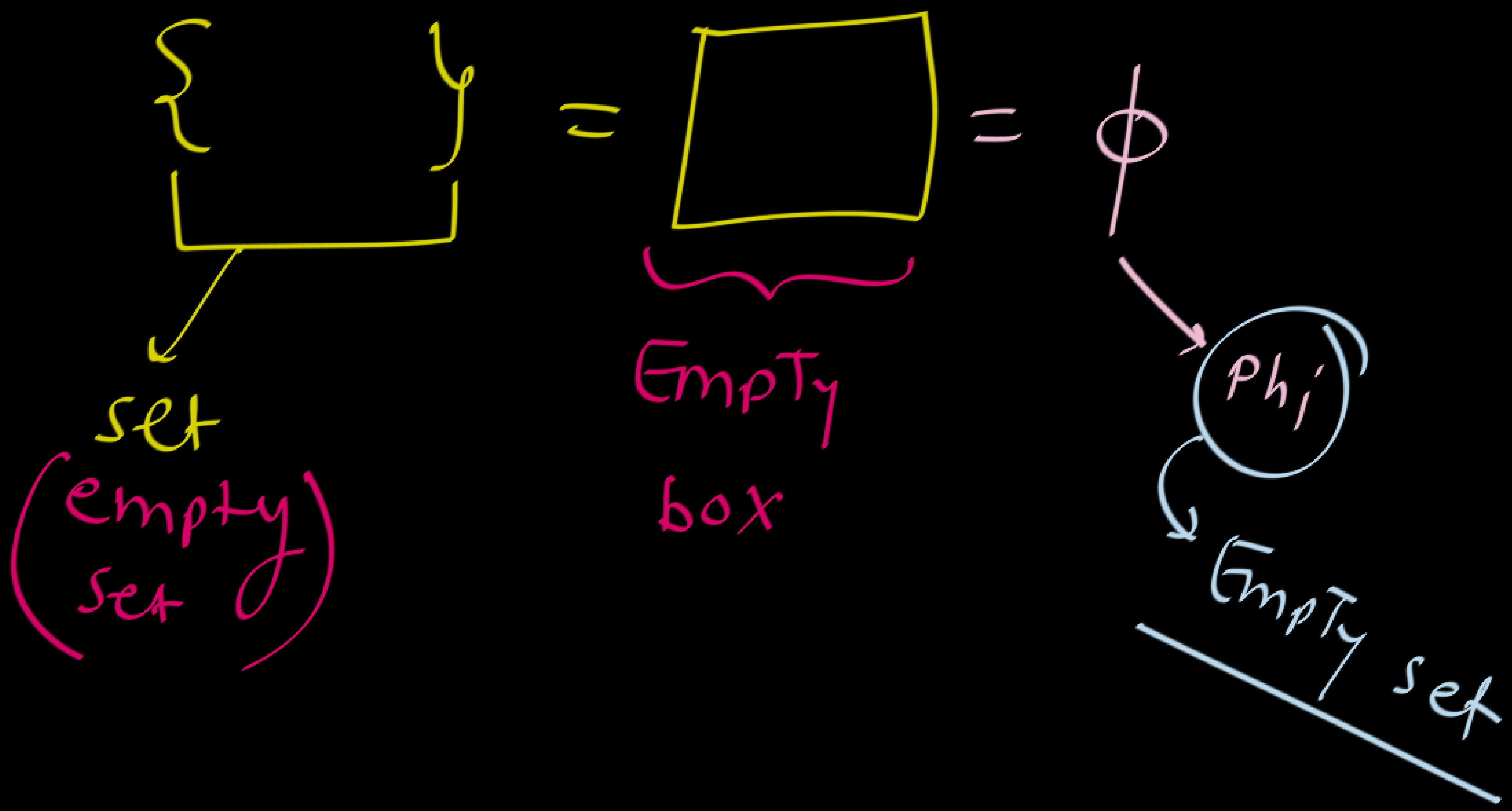


{ } = □ = φ
set
(empty set)
Empty box
Phi
Empty set

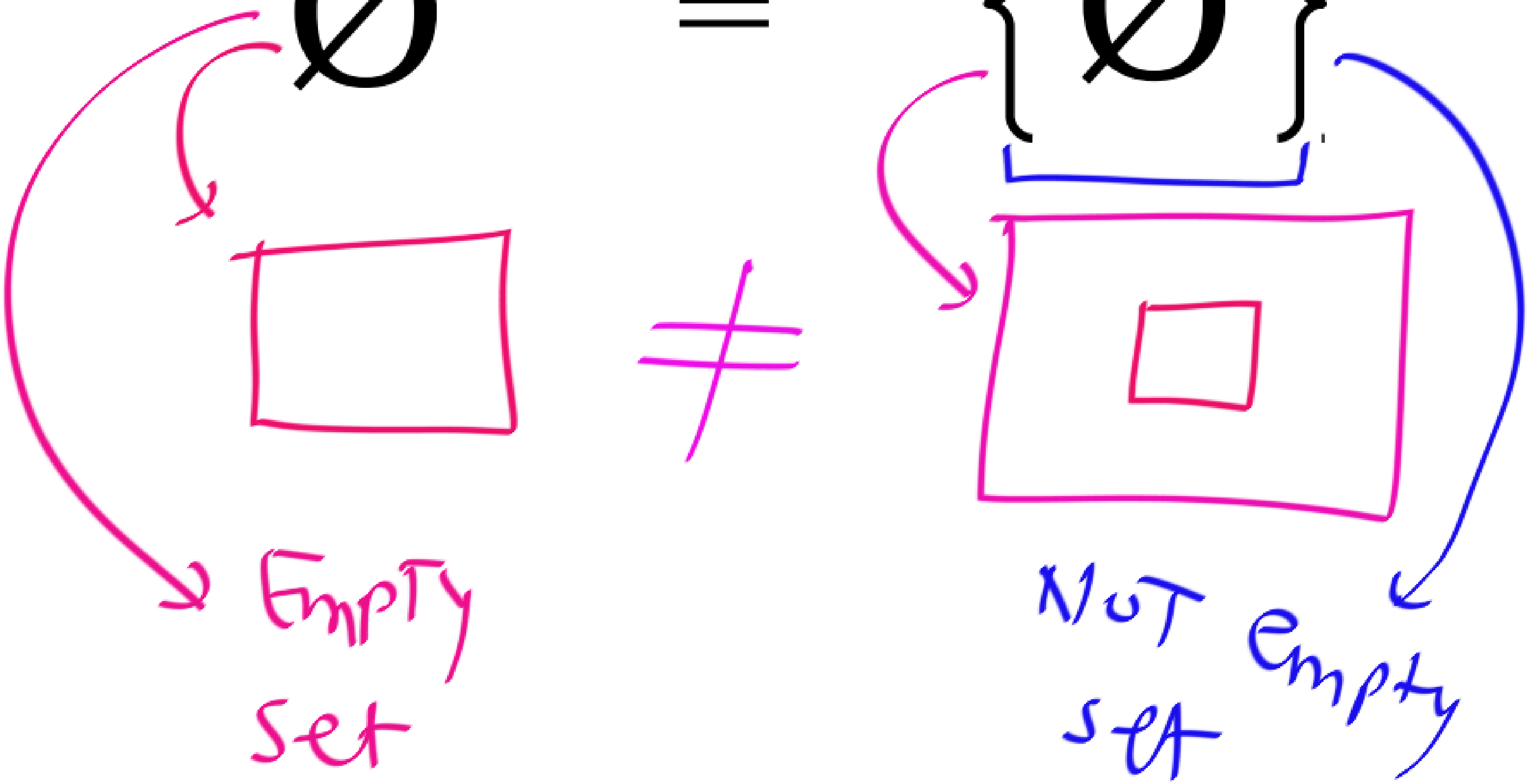
Definition 2.1 *The* **empty set** *is a set containing no objects. It is written as a pair of curly braces with nothing inside* $\{\}$ *or by using the symbol* $\emptyset$*.*

As we shall see, the empty set is a handy object. It is also quite strange. The set of all humans that weigh at least eight tons, for example, is the empty

$$\emptyset \overset{?}{=} \{\emptyset\}$$

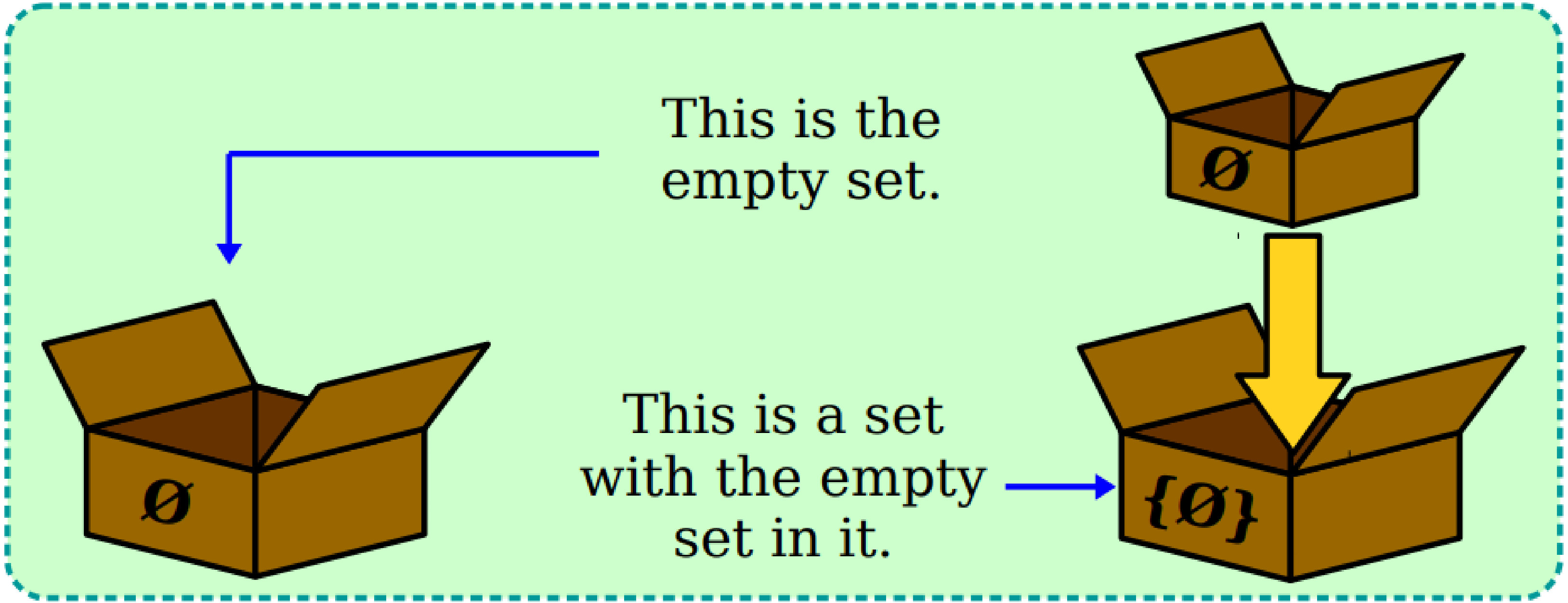
Question: Are these objects equal?

$$\emptyset \stackrel{?}{=} \{\emptyset\}$$



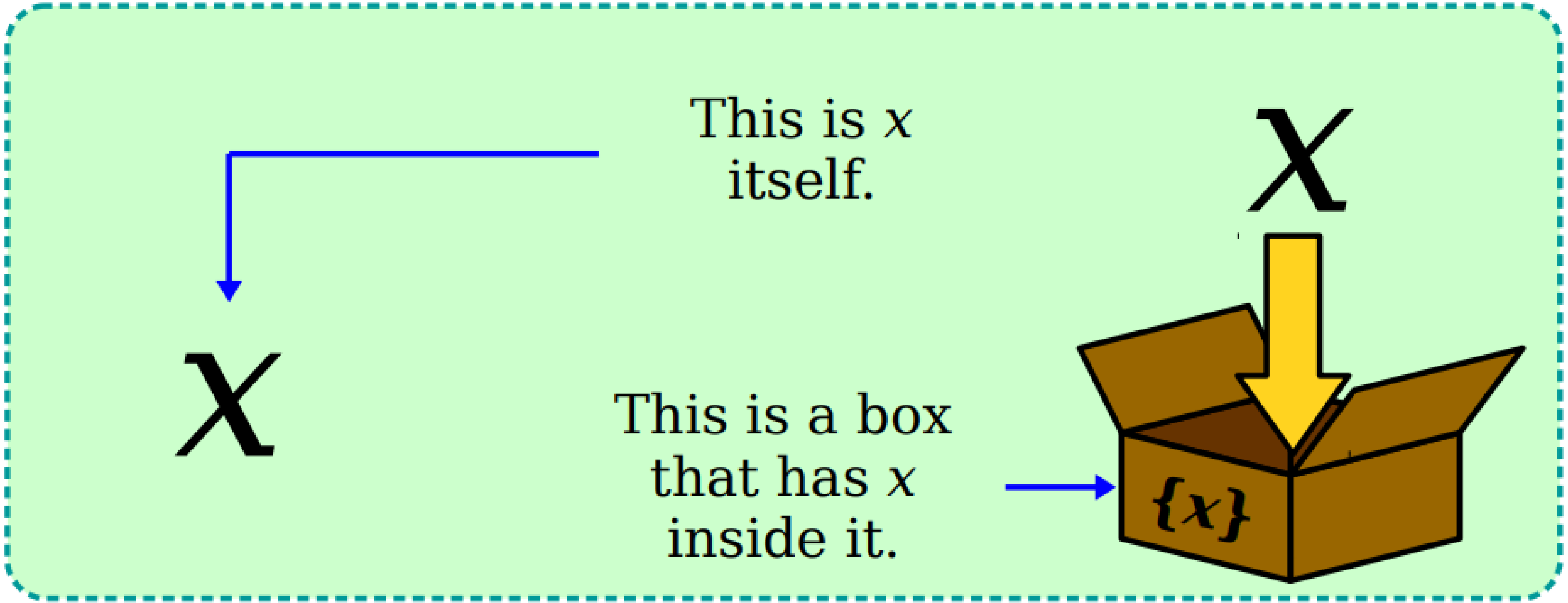
Question: Are these objects equal?

$$\emptyset \neq \{\emptyset\}$$



Question: Are these objects equal?

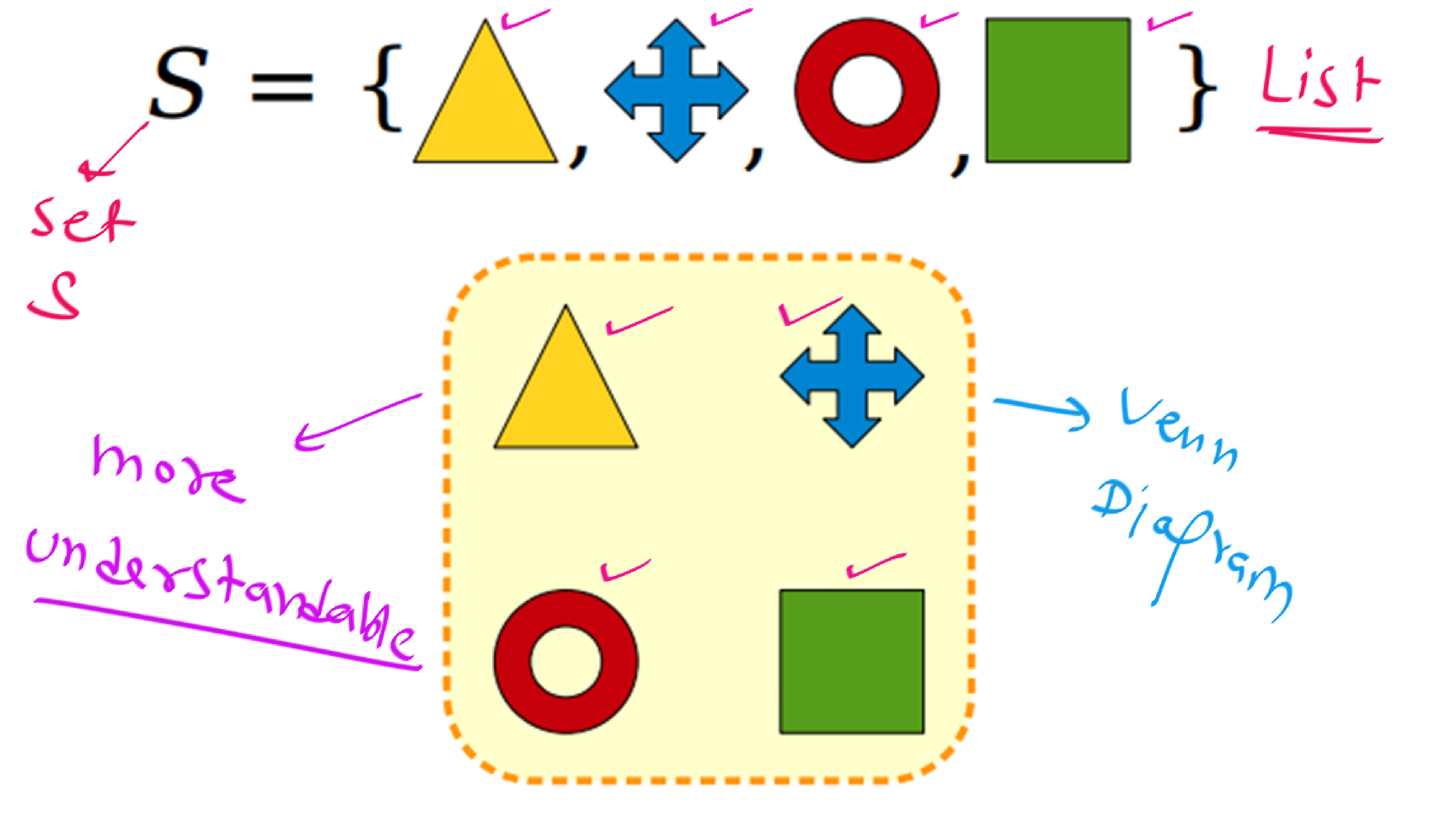
$$x \neq \{x\}$$



No object x is equal to the set containing x.

Set Representations

1. List
2. Venn Diagram
3. Set Builder Notation


S = {
,
,
,
}
List
set
S
more
understandable
Venn
Diagram

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

List Representation

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even } \}$$

The set of all n

where

n is a natural number

and n is even

$$\{$$ [redacted] $$2, 4, 6, 8, 10, 12, 14, 16, \ldots \}$$

Set Builder Notation

- A set may be specified in **set-builder notation**:

  $\{ x \mid \textit{some property x satisfies} \}$

- For example:

  $\{ r \mid r \in \mathbb{R} \text{ and } r < 137 \}$

  $\{ n \mid n \text{ is a power of two} \}$

  $\{ x \mid x \text{ is a set of US currency} \}$

  $\{ p \mid p \text{ is a legal Java program} \}$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\, x \mid \textit{some property x satisfies} \,\}$$

$$\{\, x \in S \mid \textit{some property x satisfies} \,\}$$

- For example:

$$\{\, n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \,\}$$

$$\{\, C \mid C \text{ is a set of US currency} \,\}$$

$$\{\, r \in \mathbb{R} \mid r < 3 \,\}$$ → $(-\infty, 3)$

$$\{\, n \in \mathbb{N} \mid n < 3 \,\}$$ **(the set {[redacted]1, 2})**

$x \in \{\ \}$

No such x

$S = \{x \mid x \in \phi\} = \{\ \} = \phi$

Example:

- $|\{x \mid -2 < x < 5, x \in Z\}| =?$
- $|\emptyset| =?$
- $|\{x \mid x \in \emptyset \text{ and } x < 3\}| =?$
- $|\{x \mid x \in \{\emptyset\}\}| =?$

$\to 6$

Example:

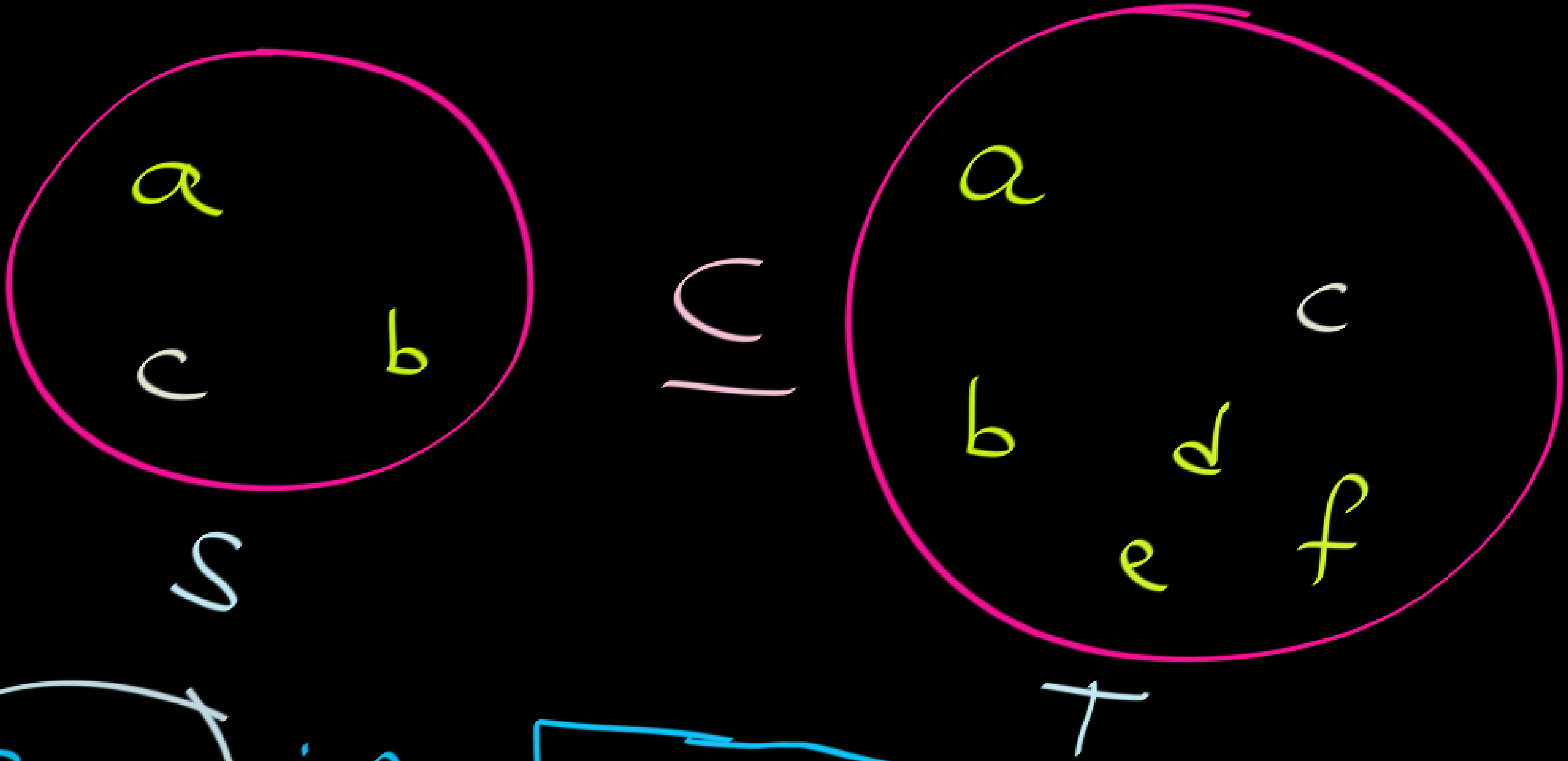
- $|\{x \mid -2 < x < 5, x \in Z\}| =?$ $= |\{-1, 0, 1, 2, 3, 4\}|$
- $|\emptyset| =?$ $= 0$
- $|\{x \mid x \in \emptyset \text{ and } x < 3\}| =?$ $= |\phi| = 0$
- $|\{x \mid x \in \{\emptyset\}\}| =?$ $= |\{\phi\}| = 1$

$\phi = \{\ \} \to |\phi| = |\{\ \}| = 0$

$x \in \{\phi\}$ → Not Empty set

$x = \phi$

$|\{\phi\}| = 1$



$S \subseteq T$ iff $\text{if } x \in S \text{ then } x \in T$

Subsets

- A set S is called a ***subset*** of a set T (denoted $S \subseteq T$) if all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ b, c \} \subseteq \{ a, b, c, d \}$
  - $\{ \text{H}, \text{He}, \text{Li} \} \subseteq \{ \text{H}, \text{He}, \text{Li} \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Subset

Definition:

A set A is a subset of B if every element of A is also in B.

- We write $A \subseteq B$ if A is a subset of B.
- Clearly, for any set A, the empty set $\emptyset$ (which does not contain any element) and A itself are both subsets of A.

Definition:

If $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, and we write $A \subset B$.

$\{1, 2\} \subseteq \{1, 2\}$

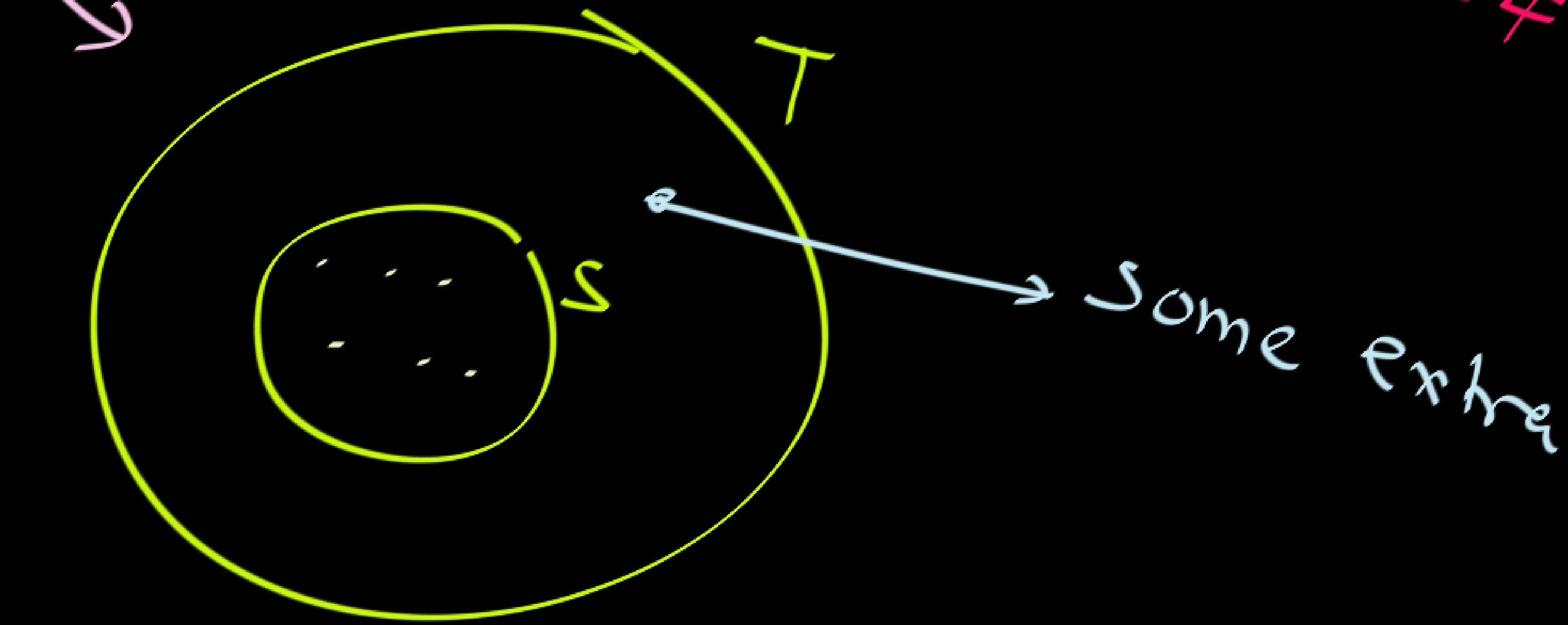
$\{1, 2\} \not\subseteq \{1\}$

$\{1, 2\} \not\subseteq \{1, 3\} \Rightarrow$ $x = 2$

Counter Example:

$x = 2$

$S \subset T$ iff $S \subseteq T$ & $S \neq T$



Subsets and Elements

- We say that $\boldsymbol{S \in T}$ if, among the elements of T, one of them is *exactly* the object S.
- We say that $\boldsymbol{S \subseteq T}$ if S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true.)
  - Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.

Note:

for Any (→ Every) set S:

1. $S \subseteq S$ ✓
2. $\phi \subseteq S$

Proper Subsets

- By definition, any set is a subset of itself. *(Why?)*
- A **proper subset** of a set S is a set T such that
  - $T \subseteq S$
  - $T \neq S$
- There are multiple notations for this; they all mean the same thing:
  - $T \subsetneq S$
  - $T \subset S$

Definition 2.12 *For two sets S and T we say that S* **is a subset of** *T if each element of S is also an element of T. In formal notation $S \subseteq T$ if for all $x \in S$ we have $x \in T$.*

If $S \subseteq T$ then we also say T contains S which can be written $T \supseteq S$. If $S \subseteq T$ and $S \neq T$ then we write $S \subset T$ and we say S is a *proper* subset of T.

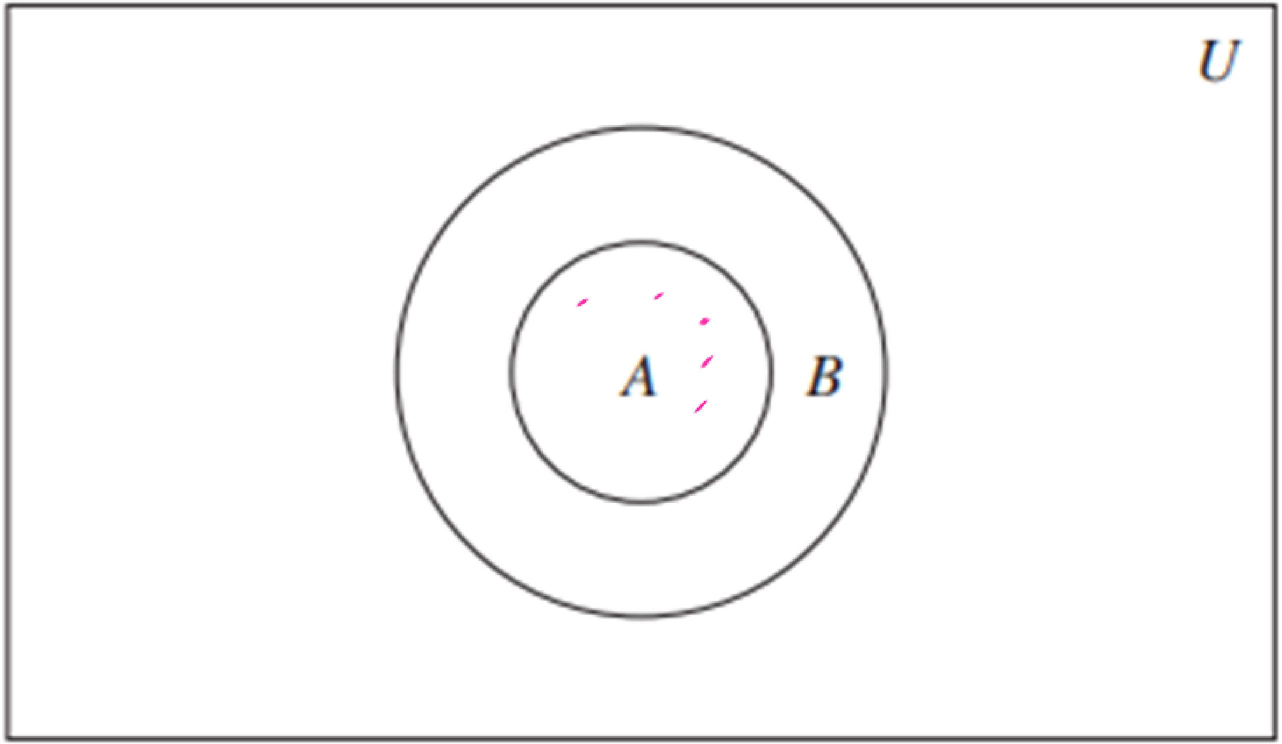


FIGURE 2 Venn Diagram Showing that A Is a Subset of B.

$S = \{1, 2\}$

Subsets of S ; → SubCollections

$\{\ \}, \{2\}, \{1\}, \{1, 2\}$

$S = \{1, 2\}$

Powerset of $S \equiv P(S) \equiv 2^S$:

$\{\{\ \}, \{2\}, \{1\}, \{1,2\}\}$

Example 2.9 Subsets

If $A = \{a, b, c\}$ *then A has eight different subsets:*

$$\emptyset \qquad \{a\} \qquad \{b\} \qquad \{c\}$$

$$\{a, b\} \qquad \{a, c\} \qquad \{b, c\} \qquad \{a, b, c\}$$

Notice that $A \subseteq A$ *and in fact each set is a subset of itself. The empty set* $\emptyset$ *is a subset of every set.*

Note: a finite set S of n elements

subsets of S : 2^n

$S = \{a, b, c\}$ (2, 2, 2)

→ subset $S_1 = \{ \quad \}$

every element of S has 2 choices

Note: a finite set S of n elements

subsets of S : $2^n - 1$

↓ proper

→ for S itself

Q: $S = \{a, b\}$

which is a proper subset of S?

(1) $\{a\}$ ✓

(2) $\{a, b\}$ ✗

(3) ϕ ✓

(4) $\{a, d\}$ ✗

(5) $\{e\}$ ✗

Q: $S = \phi$

which is a proper subset of S?

① $\{a\}$

② $\{a, b\}$

③ ϕ

④ None

Q: $S = \phi$

which is a proper subset of S?

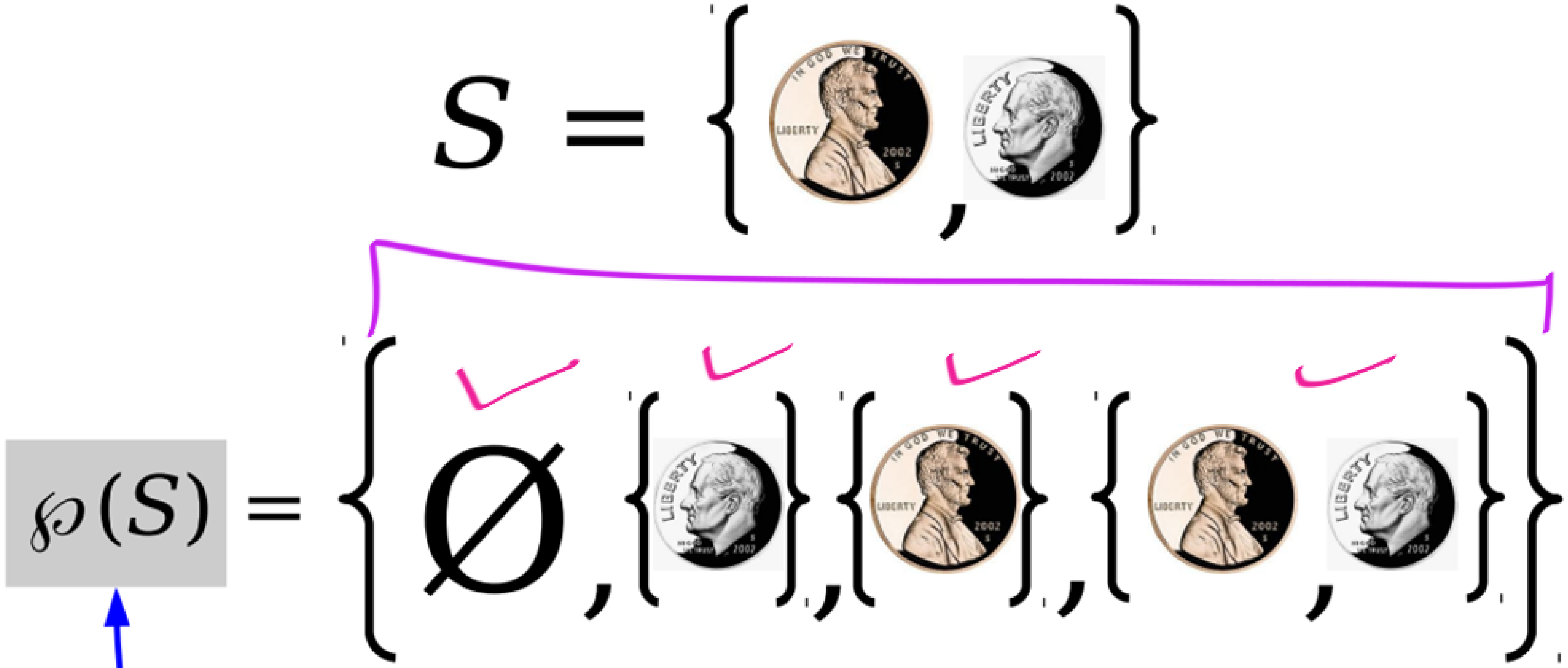
1. $\{a\}$
2. $\{a, b\}$
3. ϕ
4. None

$S \subset T$ iff

$S \neq T$ & $S \subseteq T$

Definition 2.13 *The set of all subsets of a set S is called the* **powerset** *of S. The notation for the powerset of S is $\mathcal{P}(S)$.*

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.



This is the ***power set*** of S, the set of all subsets of S. We write the power set of S as $\wp(S)$.

Formally, $\wp(S) = \{\ T \mid T \subseteq S\ \}$.

(Do you see why?)

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

Note:

Set S of n elements

$|P(S)|$ = # subsets of S

$= 2^n$

What is $\wp(\emptyset)$?

$\phi = \{ \}$

subsets of ϕ :

$\{\phi\}$

ϕ

$P(\phi) = \{\phi\}$

$$\phi = S \longrightarrow |S| = 0$$

$$P(S) = P(\phi) = \{\phi\}$$

$$|P(S)| = 2^{|S|} = 2^{0} = 1$$

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$*!*

What is the power set of the empty set? What is the power set of the set {Ø}?

$\{\phi\}$

$S = \{\phi\}$

$P(S) = \{\phi, \{\phi\}\}$

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

◀

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

Power set

Definition

The power set of set A is the set of all subsets of A. We denote it by $P(A)$.

Example:

- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
- $P(\emptyset) = \{\emptyset\}$.
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

Set Operations

Union

Intersection

Set Difference

Symmetric Difference

Set Complement

Set operations

Recall:

- We have $+, -, \times, \div, \ldots$ operators for numbers.
- We have $\vee, \wedge, \neg, \rightarrow \ldots$ operators for propositions.

Question:

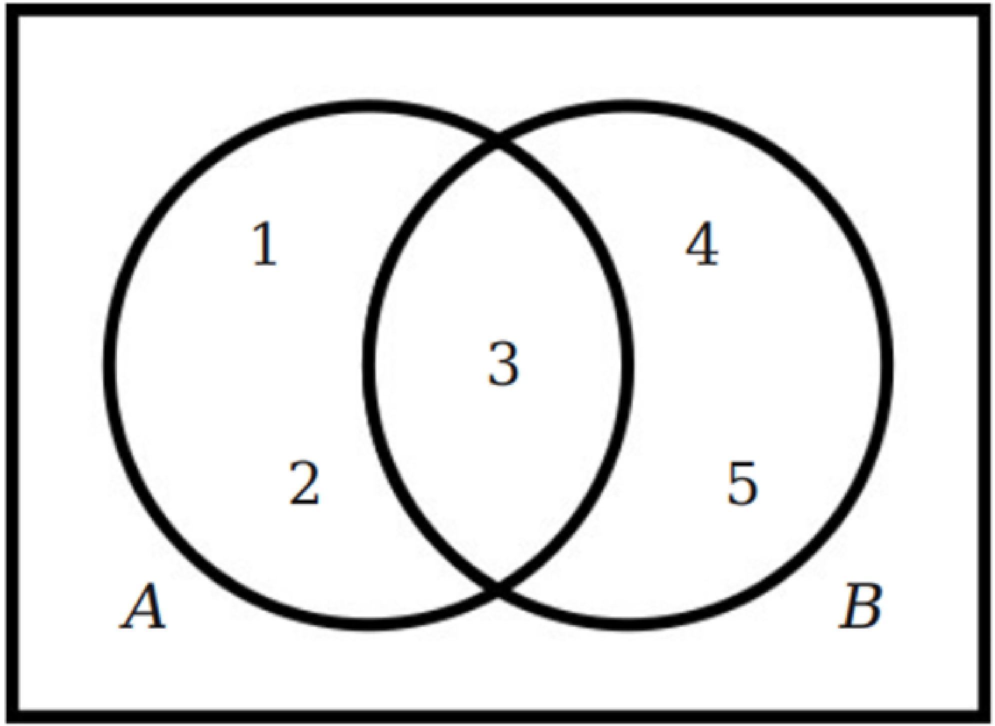
What kind of operations do we have for sets?

Answer: union, intersection, difference, complement, . . .

Universal Set and Empty Set

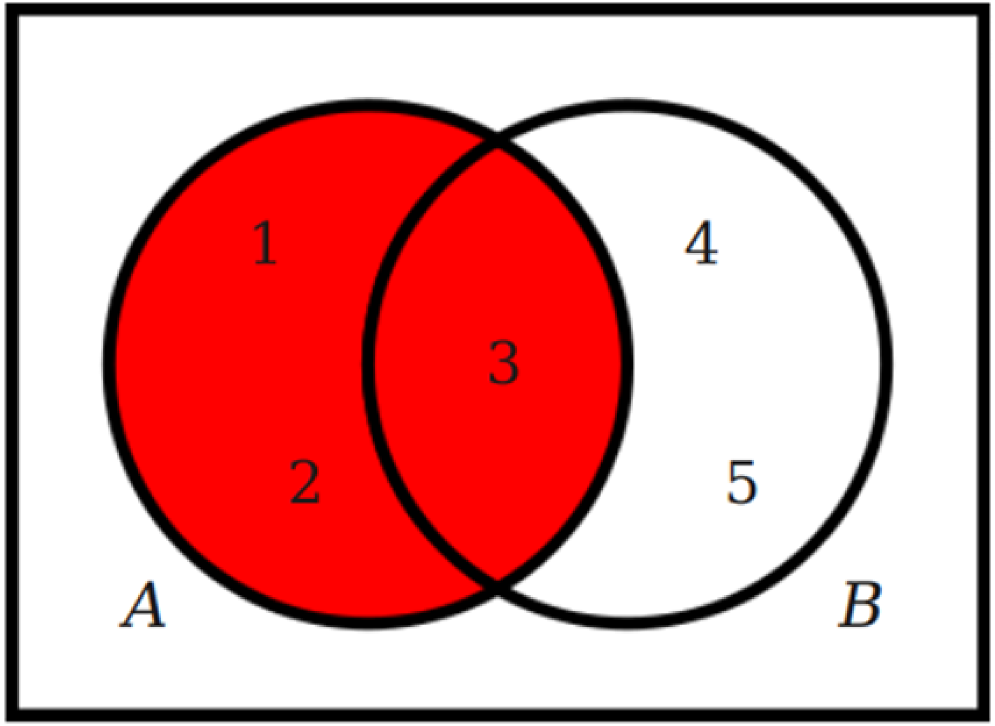
- The universal set U is the set containing everything currently under consideration.
  - Content depends on the context.
  - Sometimes explicitly stated, sometimes implicit.
- The empty set is the set with no elements. Symbolized by $\emptyset$ or $\{\}$.

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

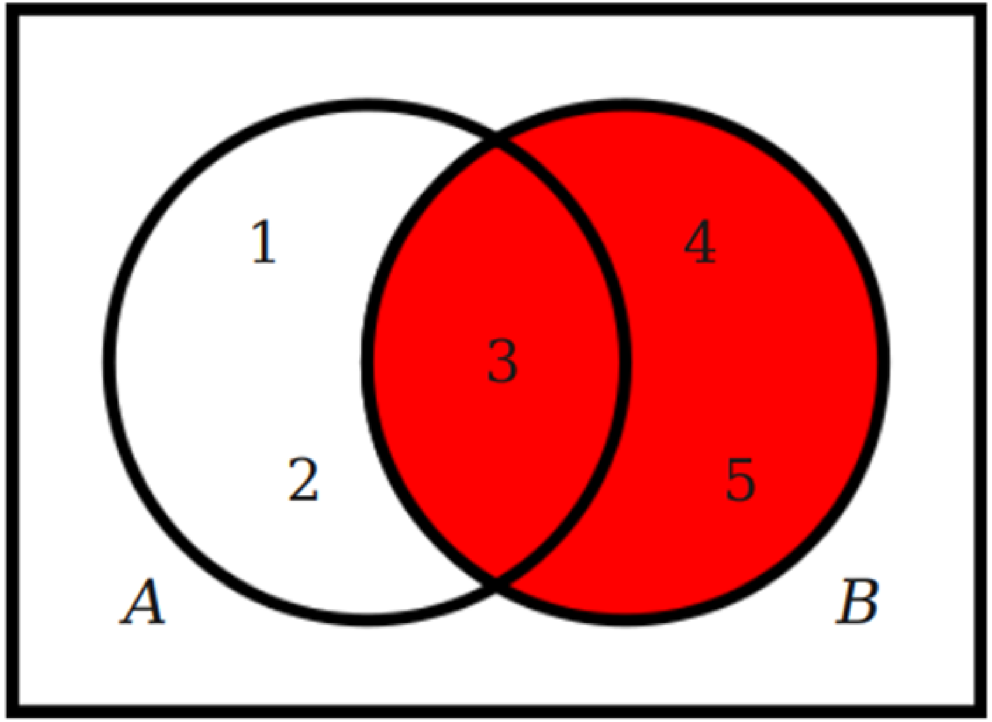
Venn Diagrams



A

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



B

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Set A, B:

$A \cup B$: Union : $\{x \mid x \in A \text{ OR } x \in B\}$

$A \cap B$: Intersection : $\{x \mid x \in A \text{ and } x \in B\}$

$A - B$: Set Difference $= \{x \mid x \in A \text{ and } x \notin B\}$

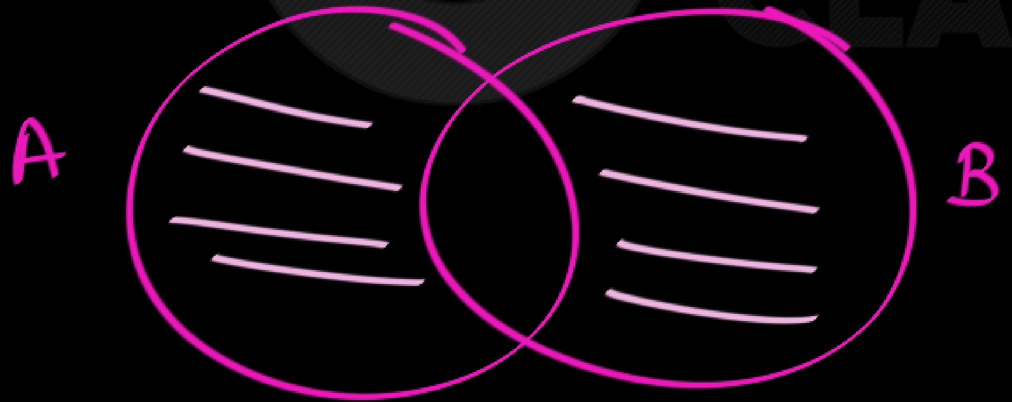
$B - A = \{x \mid x \in B \text{ AND } x \notin A\}$

- The *union* of sets A and B is the set $A \cup B = \{x : x \in A \vee x \in B\}$.
- The *intersection* of sets A and B is the set $A \cap B = \{x : x \in A \wedge x \in B\}$.
- The *set difference of* A *and* B is the set $A \setminus B = \{x : x \in A \wedge x \notin B\}$. Alternate notation: $A - B$.
- The *symmetric difference of* A *and* B is $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Note: $A \oplus B = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$.

Symmetric Difference

$\triangle$ or $\oplus$

$$A \oplus B = (A - B) \cup (B - A)$$



Symmetric Difference $\triangle$ or $\oplus$

$$A \oplus B = (A - B) \cup (B - A)$$

$A \oplus B$ = Set of Exclusive elements of A, B $= (A - B) \cup (B - A)$

$$A \oplus B = A \triangle B = \left\{ x \;\middle|\; \begin{array}{c} (x \in A \text{ and } x \notin B) \\ \text{OR} \\ (x \in B \text{ and } x \notin A) \end{array} \right\}$$

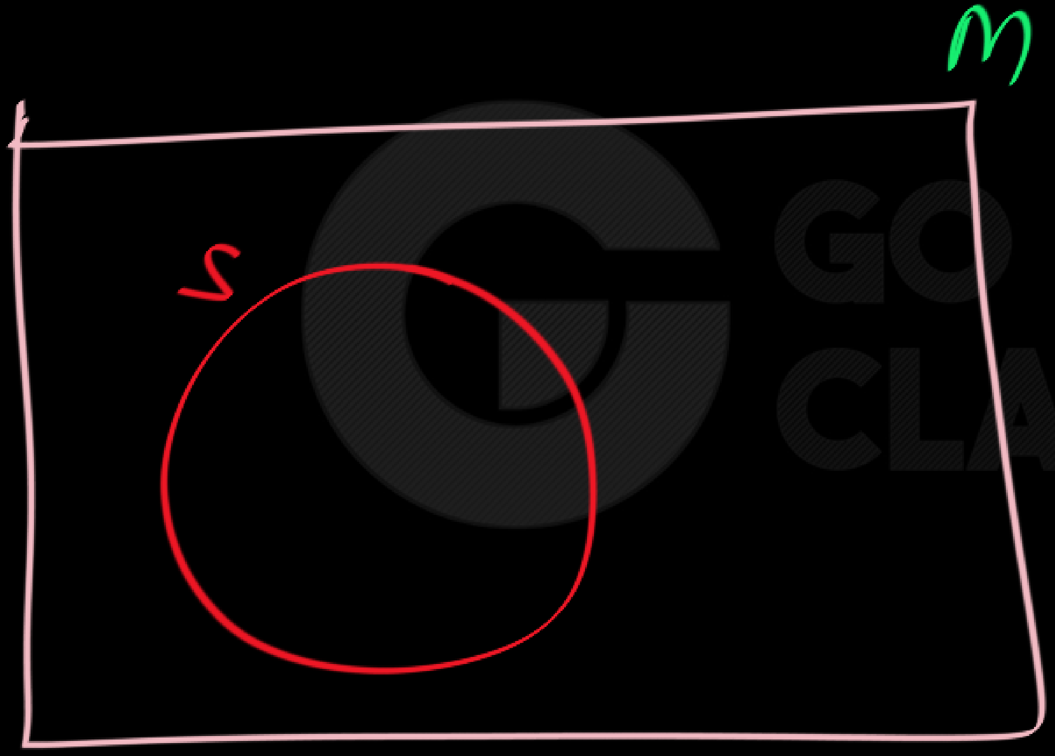
The *universe*, $\mathcal{U}$, is the collection of all objects that can occur as elements of the sets under consideration.

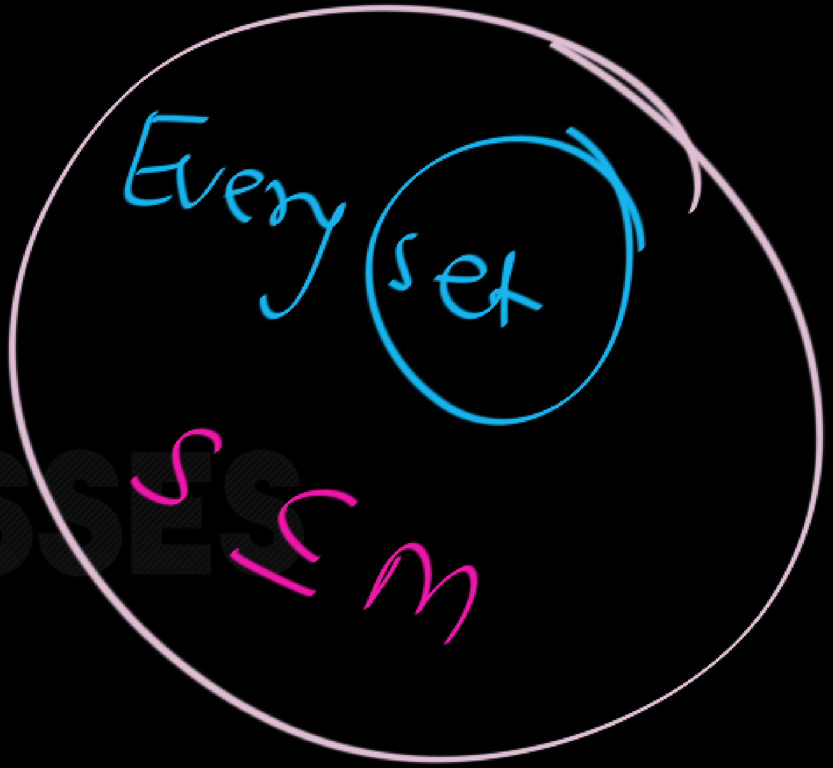
- The *complement of* A is $A^c = \mathcal{U} \setminus A = \{x : x \notin A\}$.

Set S

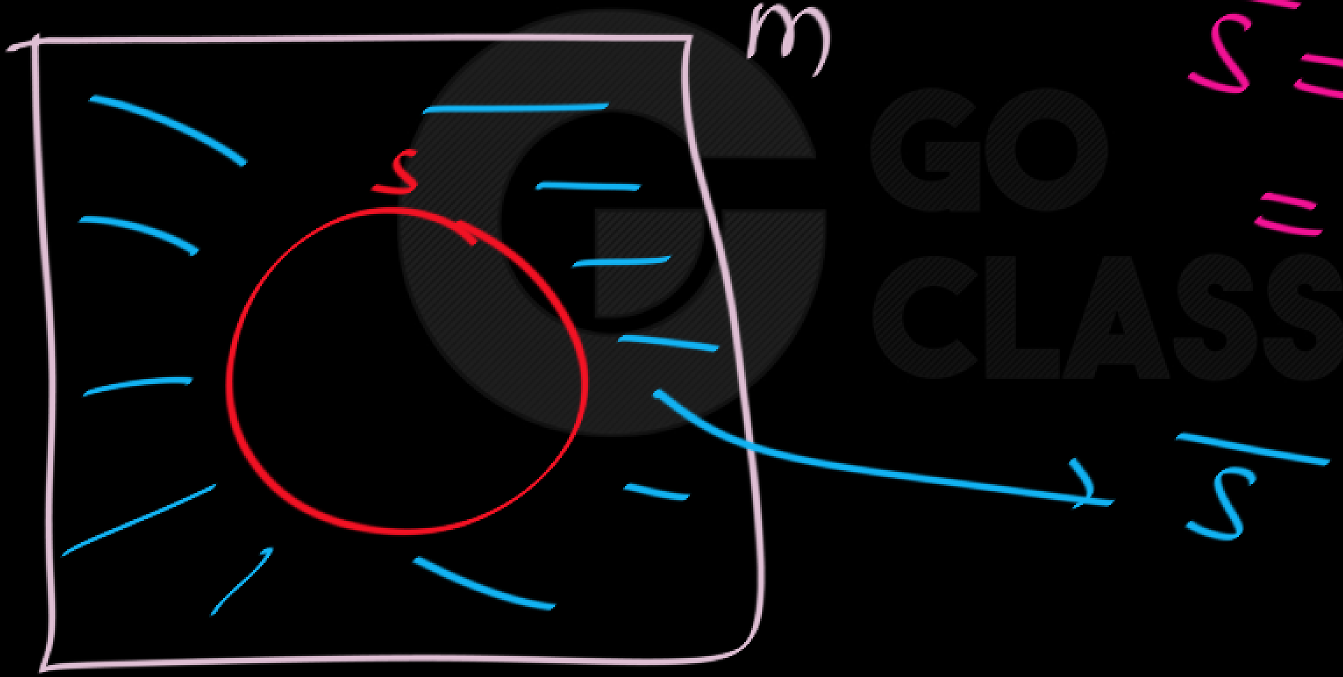
$$\overline{S} = S' = \neg S = \sim S = \{x \mid x \notin S\}$$

Universal set M:





Universal set M :



$\bar{S} = M - S$

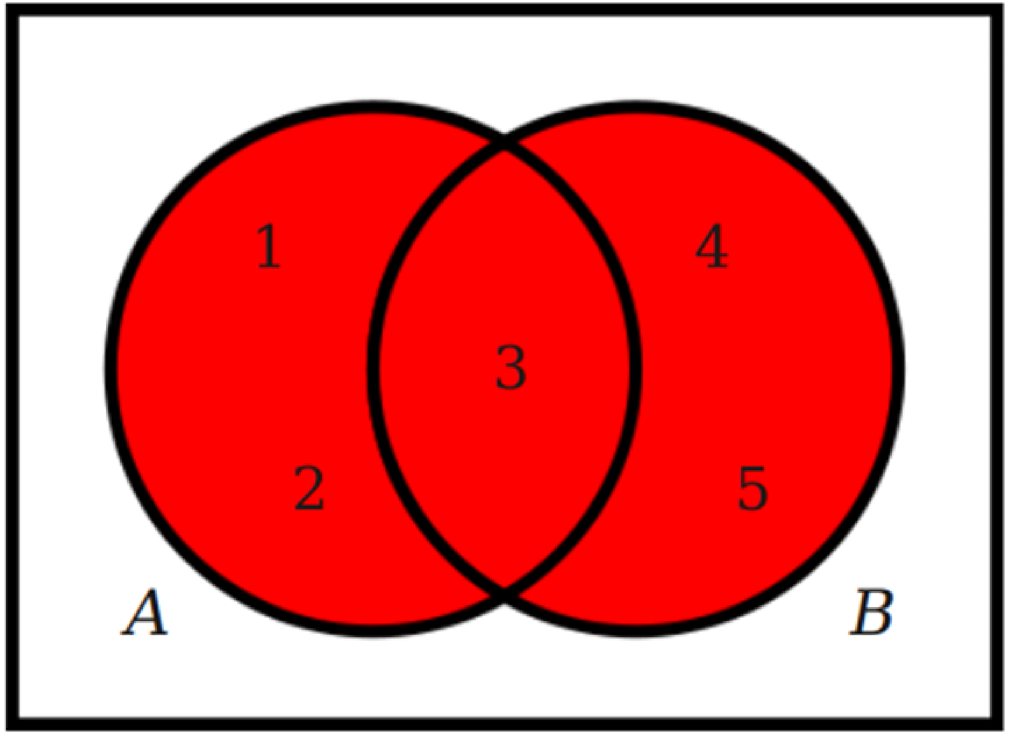
$= \{x \mid x \notin S\}$

Note:

Set Difference Symbol: $-$, $\setminus$

$A - B$ OR $A \setminus B$

Venn Diagrams



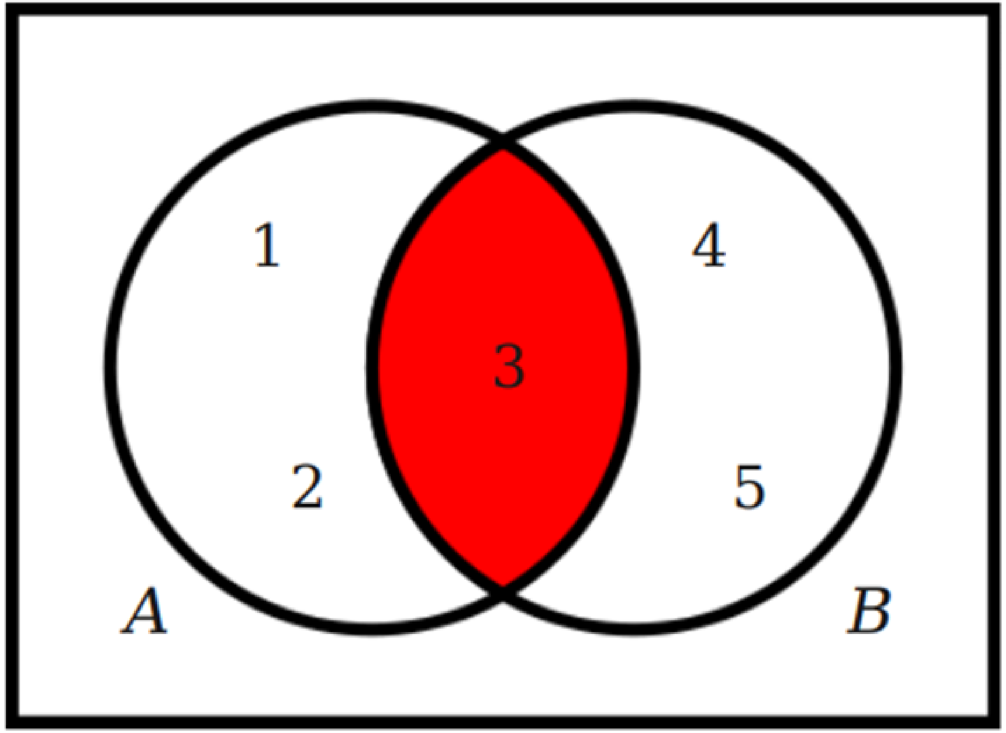
Union

$A \cup B$

$\{ 1, 2, 3, 4, 5 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



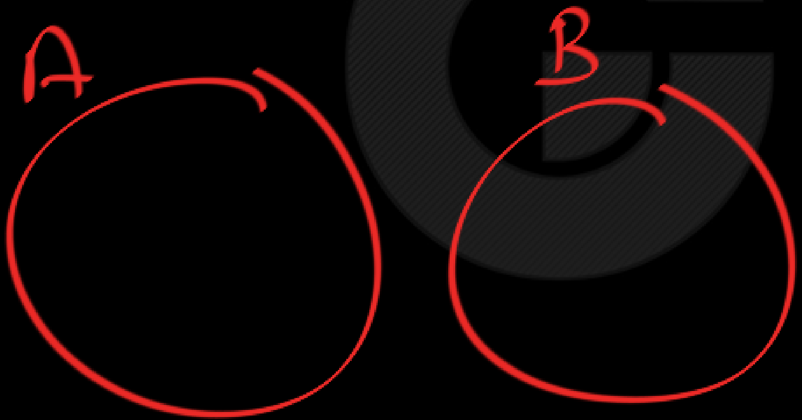
Intersection

$A \cap B$

$\{ 3 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Definition 2.6 *If A and B are sets and* $A \cap B = \emptyset$ *then we say that A and B are* disjoint, *or* disjoint sets.



$A \cap B = \phi$

then A, B are called Disjoint sets

Q:

$A = \phi$

$B = \phi$

$\{\ \}$ $\{\ \}$

Disjoint

$A \cap B = \phi$

Disjoint set

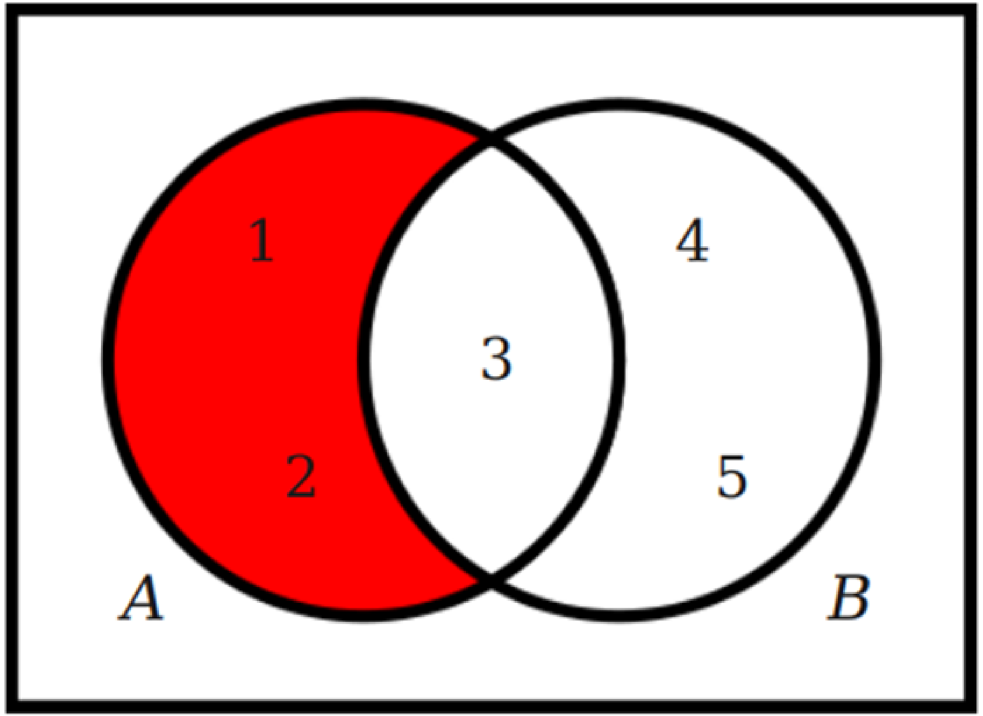
Definition:

Two sets A and B are disjoint if $A \cap B = \emptyset$.

Example:

- $\{1,2,3\} \cap \{4,5\} = \emptyset$, so they are disjoint.
- $\{1,2,3\} \cap \{3,4,5\} \neq \emptyset$, so they are not disjoint.
- $Q \cap R^+ \neq \emptyset$, so they are not disjoint.
- $\{x \mid x < -2\} \cap R^+ = \emptyset$, so they are disjoint.

Venn Diagrams



Difference

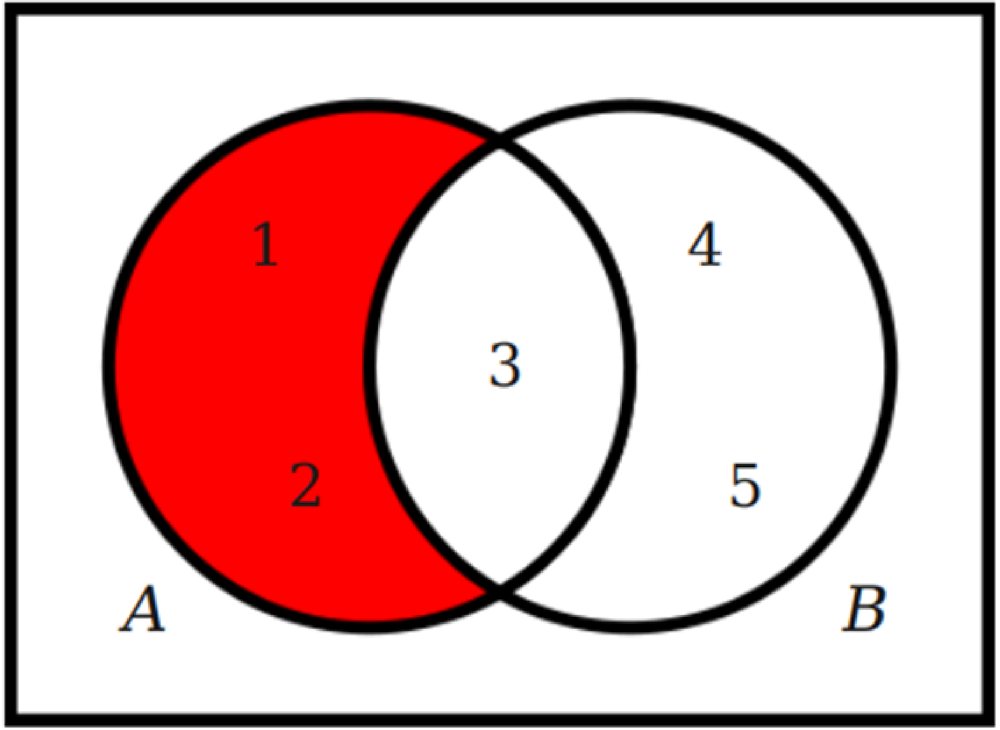
$A - B$

{ 1, 2 }

$A \setminus B$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

$A \setminus B$

{ 1, 2 }

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

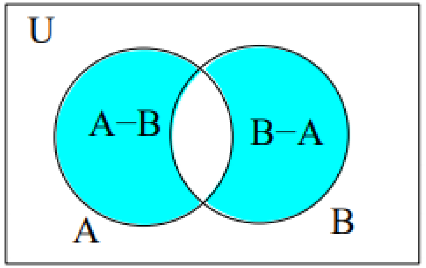
Symmetric Difference

Definition:

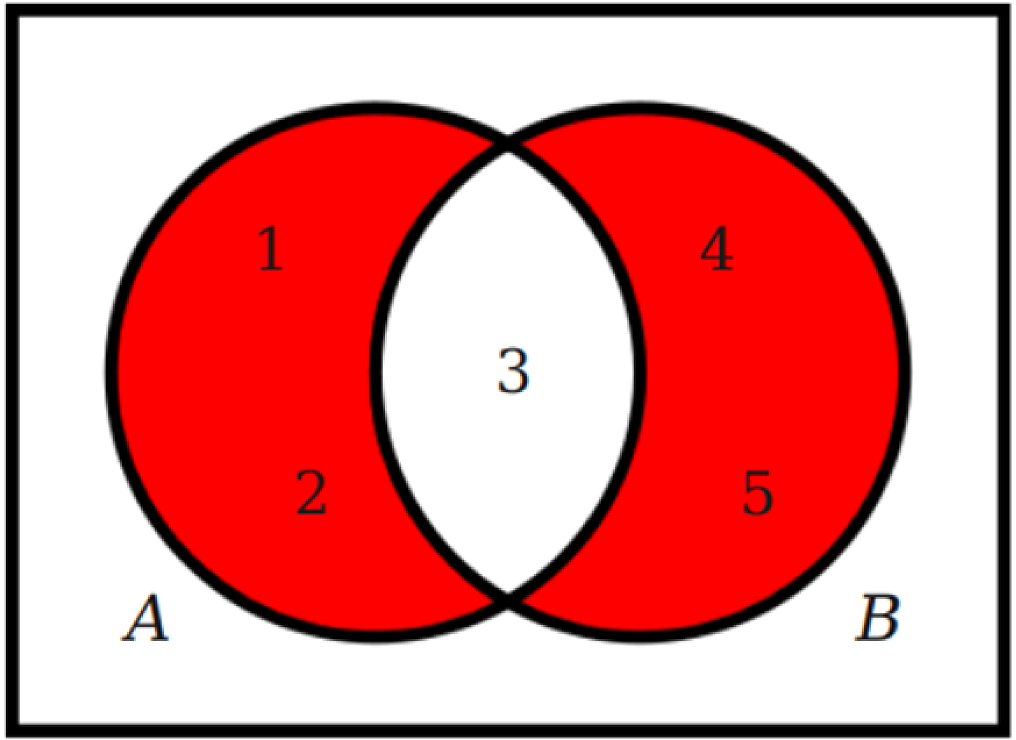
The symmetric difference of set A and set B, denoted by $A \oplus B$, is the set containing those elements in exactly one of A and B.

- Formally: $A \oplus B = (A - B) \cup (B - A)$.

Venn Diagram of Symmetric Difference Operation:



Venn Diagrams



Symmetric Difference

$A \,\Delta\, B$

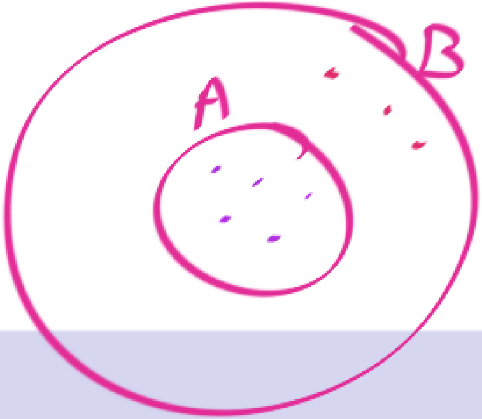
$\{\ 1, 2, 4, 5\ \}$

$A = \{\ 1, 2, 3\ \}$
$B = \{\ 3, 4, 5\ \}$

Example for calculating set operations

$A \subseteq B$

$A \subset B$



Example:

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then:

- $A \cap B = \{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A - B = \emptyset$
- $B - A = \{6, 7, 8\}$

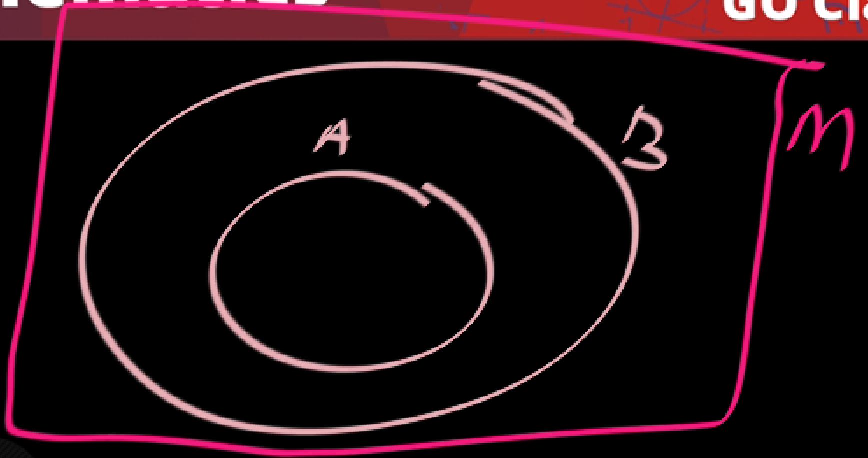
Note: Set A, B.

If $A \subseteq B$ then

1. $A \cup B$
2. $A \cap B$
3. $B - A$
4. $A - B$
5. $A \oplus B$
6. $\overline{A}$
7. $\overline{B}$

Note: Set A, B.

If $A \subseteq B$ then



① $A \cup B = B$

② $A \cap B = A$

③ $B - A = B - A$

④ $A - B = \phi$

⑤ $A \oplus B = B - A$

⑥ $\overline{A} = m - A$

⑦ $\overline{B} = m - B$

Definition:

Two sets are equal if and only if they have the same elements.

- Note that the order of elements is not a concern since sets do not specify orders of elements.
- We write $A = B$, if A and B are equal sets.

Example:

- $\{1,2,3\} = \{2,1,3\}$
- $\{1, 2, 3, 4\} = \{x \in Z$ and $1 \leq x < 5\}$

Note:

Set A, B

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

Q:

$A = \{1, 2, 3\}$

$B = \{1, 1, 2, 2, 2, 3, 3\}$

$A = B$?

Q:

$A = \{1, 2, 3\}$

$B = \{1, 1, 2, 2, 2, 3, 3\} = \{1, 2, 3\}$

$A = B$? YES

$A \subseteq B$

$B \subseteq A$

cm1

Note:

Definition Set $A \subseteq B$

$A \subseteq B$ iff If $x \in A$ then $x \in B$

Arbitrary element

Note:

Prove that set $A \subseteq B$

Take Arbitrary (Random) element $x \in A$
(Assume)
then Show that

$x \in B$

Note:

Prove that set $B \subseteq A$

Take Arbitrary (Random) element $x \in B$
(Assume)

then Show that

$x \in A$

Note:

Prove that Set $A = B$

(1) Prove $A \subseteq B$

and

(2) Prove $B \subseteq A$

NOTE:

Most of Set Theory Questions

→ Easily Solvable using Venn Diagrams

Fact:

Suppose A and B are sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

This fact is often used to prove set identities.

How to prove set identities

To prove a set identity:

$$X = Y$$

it is necessary and sufficient to show two things:

- $X \subseteq Y$ and
- $Y \subseteq X$

Equivalently, it is necessary and sufficient to show two things:

- $x \in X \to x \in Y$ and
- $x \in Y \to x \in X$

Q: Is $A, \bar{A}$ Always Disjoint ?

Q: Is $A, \overline{A}$ Always Disjoint?

$A \cap \overline{A} = \phi$

YES

Note: $A = \phi$, $\overline{A} = U - A$ (Universal set)

Q: universal set $= \phi$

$A = \phi$

$A, \bar{A}$ are Disjoint ?

Q: universal set $= \phi$

$A = \phi$

$A, \bar{A}$ are Disjoint? YES

$A \cap B = \phi$

$A = \phi$

$\bar{A} = \phi$

$\Rightarrow A \cap \bar{A} = \phi$

Set Identities

- Identity laws

$$A \cup \emptyset = A \quad A \cap U = A$$

- Domination laws

$$A \cup U = U \quad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \quad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

- Complement laws

$$A \cap \overline{A} = \emptyset \quad A \cup \overline{A} = U$$

$A \cap \phi = \phi$

$A \cup \phi = A$

$A - \phi = A$

$\phi - A = \phi$

$\phi \subseteq A$

$A \oplus \phi = A$

$\overline{\phi} =$ universal set

$\overline{(\overline{A})} = A$ ✓

$A \cup A = A$; $A \cap A = A$

$A \cup A = A$

$A \cap A = A$

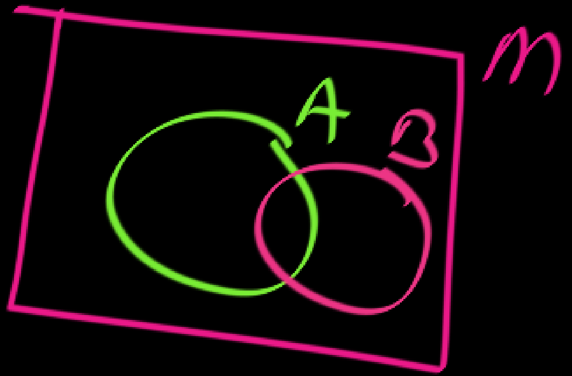
$A - A = \phi$

$A \oplus A = \phi$

$A \cap \bar{A} = \phi$

$A \cup \bar{A} =$ universal set

Universal set M



① Set $A \subseteq M$

② $A \cap M = A$

$A \cup M = M$

$A - M = \phi$

$M - A = \overline{A}$

$\overline{M} = \phi$

$\overline{\phi} = M$

$M \oplus A = M - A$

③ $\cup, \cap$ are Associative

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$= A \cup B \cup C$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$= A \cap B \cap C$$

Set operations Associative OR NOT?

1. $\cup$
2. $\cap$
3. $-$
4. $\oplus$

Set operations Associative OR NOT ?

(1) ∪ ✓

(2) ∩ ✓

(3) − ✗

(4) ⊕ ✓

(3) $\cup$, $\cap$ are Commutative:

$A \cup B = B \cup A$ ✓

$A \cap B = B \cap A$ ✓

$A - B \neq B - A$

$A \oplus B = B \oplus A$ ✓

Q: If $A - B = B - A$ then ?

If $A - B = B - A$ then

$A = B$ AND Converse

Note: $A = B$ iff $A - B = B - A$

If $A - B = B - A$ then $A = B$ AND Converse

Note: $A \neq B$ iff $A - B \neq B - A$

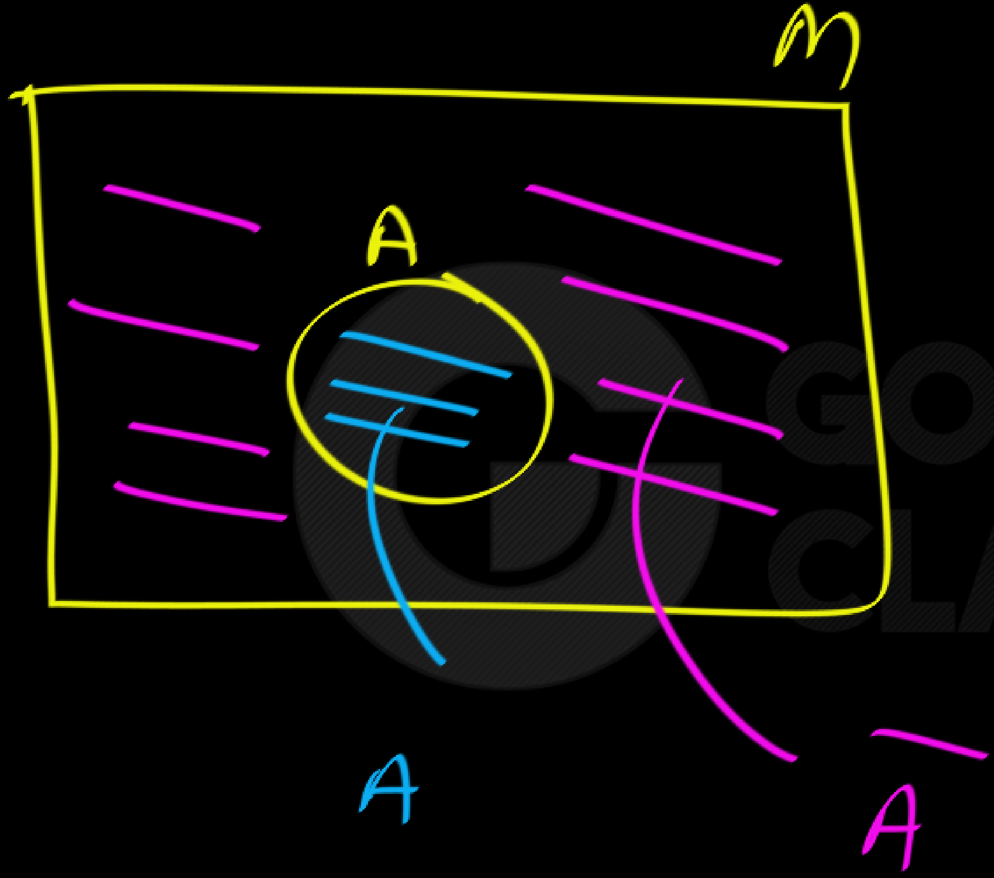
If $A \neq B$

$A - B$ = Exclusive in A
$B - A$ = " " B

} can NOT be same.

If $A \neq B$

then $A \oplus B \neq \phi$



$A \cap \bar{A} = \phi$

$A \cup \bar{A} = M$

If $x \in A$ then $x \notin \overline{A}$

If $x \in \overline{A}$ " $x \notin A$

If $x \notin A$ then $x \in \overline{A}$

" $x \notin \overline{A}$ " $x \in A$

Set Identities (cont.)

- Commutative laws

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

- Associative laws
  $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption laws

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

∪ is Distributive over ∩

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$\cap$ is Distributive over $\cup$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

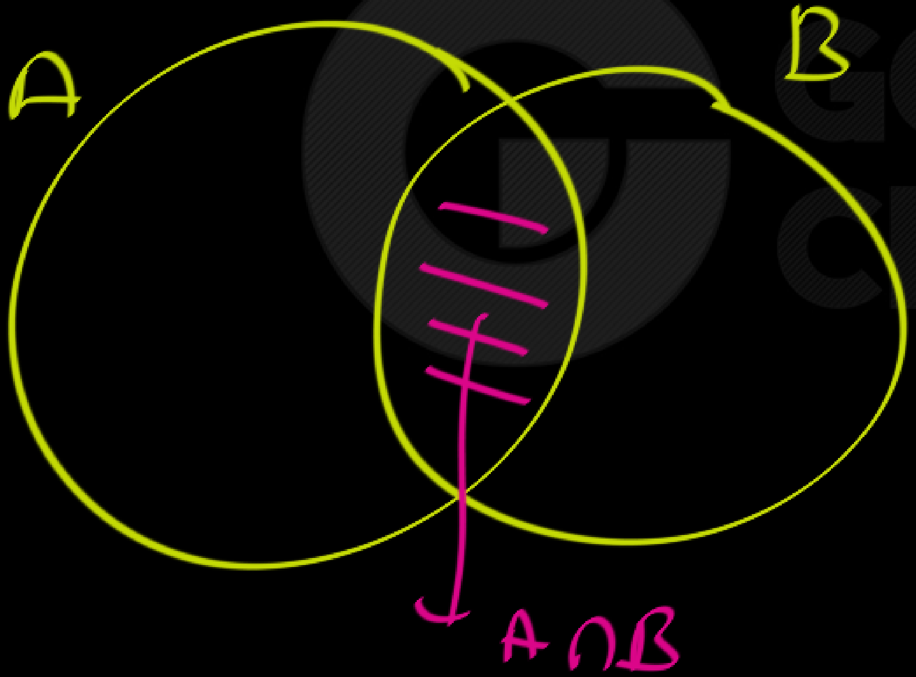
Very Imp:

① $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ ✓

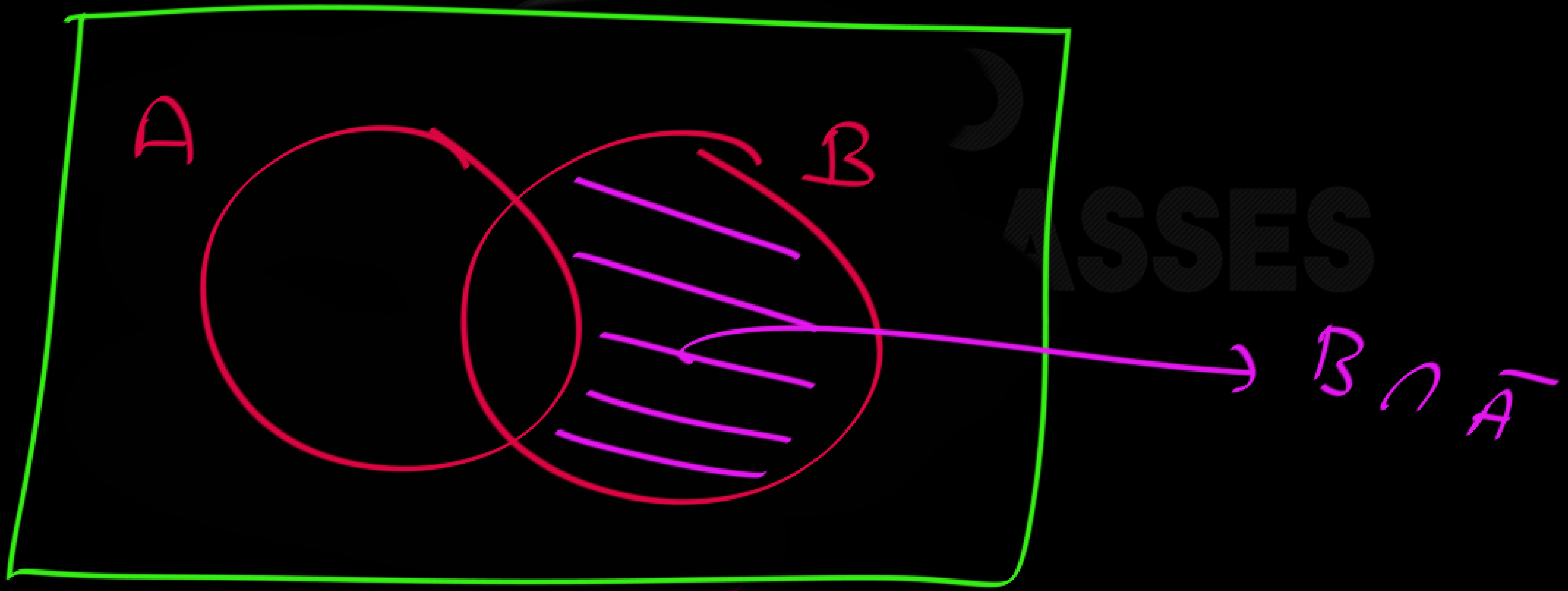
$\overline{(A \cup C)} = \overline{A} \cap \overline{C}$ ✓

② $A \cup (A \cap C) = A$ ✓

$A \cup (A \cap B) = A$



$$(\overline{A} \cap B) = B - A$$



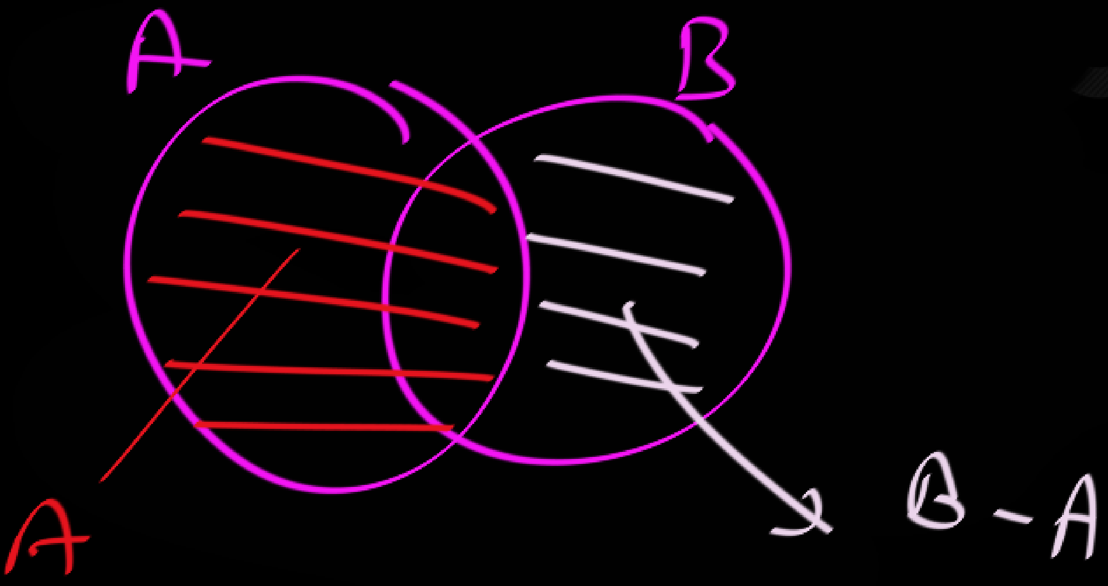
Imp:

$$A - B = A \cap \overline{B}$$

$$B - A = B \cap \overline{A}$$

$$A \cup (\overline{A} \cap B) = A \cup (B - A)$$

$$= A \cup B$$



$$A \cup (\bar{A} \cap B) = A \cup B$$

Set Theory

Prop. logic:

$$p + \bar{p}q = p + q$$

$$p + pq = p$$

Set

$$A \cup (\bar{A} \cap B) = A \cup B$$

$$A \cup (A \cap B) = A$$

Set identities: DeMorgan law

DeMorgan laws:

Just like the DeMorgan laws in logic, we have:

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} \\ \overline{A \cup B} &= \overline{A} \cap \overline{B} \end{aligned}$$

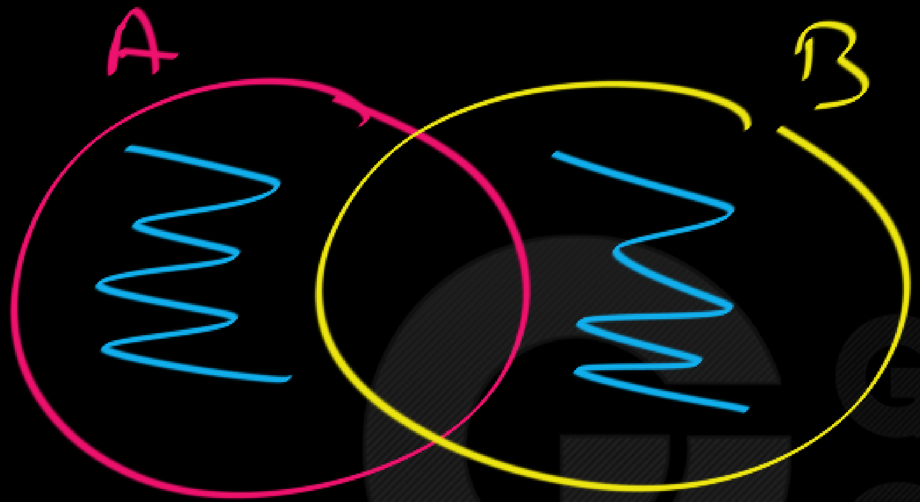
Example:

Let the universe be {0,1,2,3}.
$\overline{\{0,1\} \cap \{1,2\}} = \overline{\{1\}} = \{0,2,3\} = \{2,3\} \cup \{0,3\} = \overline{\{0,1\}} \cup \overline{\{1,2\}}$.

For each Law of Logic, there is a corresponding Law of Set Theory.

- Commutative: $A \cup B = B \cup A, \quad A \cap B = B \cap A.$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  and also on the right: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A), \quad (B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
- Double Complement: $(A^c)^c = A$
- DeMorgan's Laws: $(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$
- Identity: $\emptyset \cup A = A, \quad \mathcal{U} \cap A = A$
- Idempotence: $A \cup A = A, \quad A \cap A = A$
- Dominance: $A \cup \mathcal{U} = \mathcal{U}, \quad A \cap \emptyset = \emptyset$

$A - B = A \cap \overline{B}$



$$A \oplus B = (A-B) \cup (B-A) = (A \cup B) - (A \cap B)$$

Set equalities of note:

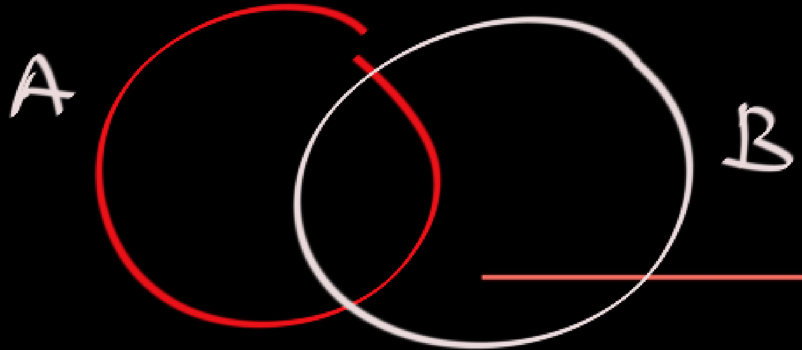
- $A \setminus B = A \cap B^c$
- $A \oplus B = (A \cup B) \setminus (A \cap B)$

Set Difference

Imp: Set A, B

① $A - B = A \cap \overline{B}$ ✓

② $A \cup (A \cap B) = A$



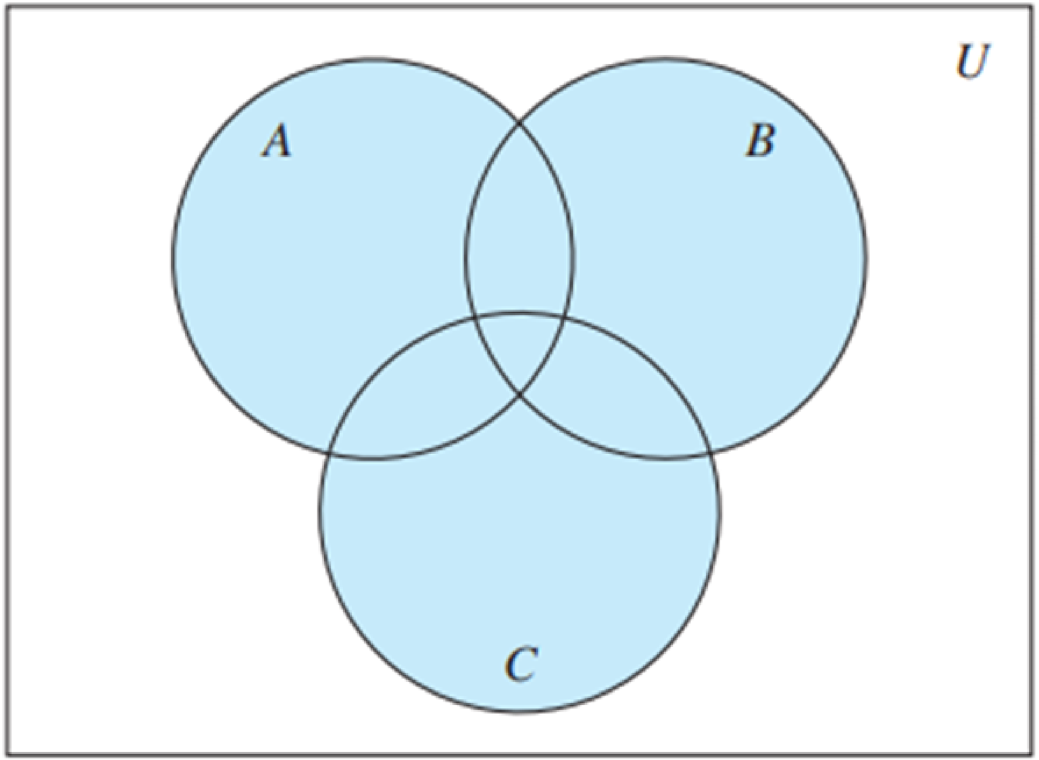
③ $A \cup (\bar{A} \cap B)$

$= A \cup B$

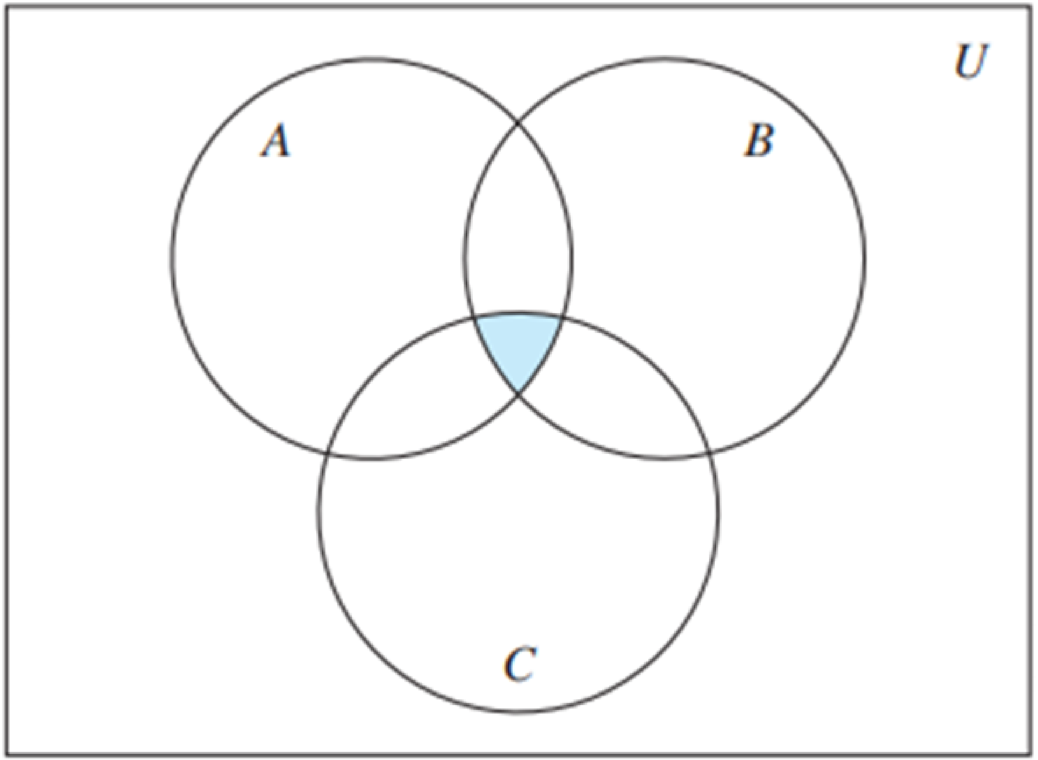
Set

many Questions

Venn Diagram



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

FIGURE 5 **The Union and Intersection of A, B, and C.**

$\{a, b\} = \{b, a\}$ **set**

$\{a, b, a\} = \{b, a\} = \{a, b\}$ set

Sequence: Order, Repetition matters

$1,1,2,2 \neq 1,1,2 \neq 2,1,1 \neq 1,1 \neq 1,2,1$

Sequence :

a, b, c ≠ b, a, c ≠ b, b, a, c

N : 1, 2, 3, ...

Tuple : → finite sequence

(1, 2) ≠ (2, 1)

2 - Tuple

n-Tuple :

$(a_1, a_2, ---, a_n)$

$(1, 1, 2) \neq (1, 2, 1)$

$(1, 1, 2) \neq (1, 2) \neq (1, 2, 1)$

3-Tuple 2-Tuple 3-Tuple

2-Tuple is called **Ordered pair**

Ordered pair: (1,1) ✓ (2,2) ✓

$(a,b) \neq (b,a)$

$(2,1) \neq (1,2)$

$(1,1) \neq (1)$

Q: $(a, b) = (b, a)$ then ?

Q: $(a, b) = (b, a)$ then ?

$a = b$

Q: $(a, b, c) = (d, e, f)$ then

$a = d$; $b = e$; $c = f$

Ordered tuple

- Recall that a set does not consider its elements order.
- But sometimes, we need to consider a sequence of elements, where the order is important.
- An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ has a_1 as its first element, a_2 as its second element, ..., a_n as its n^{th} element.
- The order of elements is important in such a tuple.
- Note that $(a_1, a_2) \neq (a_2, a_1)$ but $\{a_1, a_2\} = \{a_2, a_1\}$.

Tuples

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection of n elements, where a_1 is the first, a_2 the second, etc., and a_n the n-th (i.e., the last).
- Two n-tuples are equal iff their corresponding elements are equal.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_n = b_n$$

- 2-tuples are called ordered pairs.

Definitions

A **sequence** of objects is a list of these objects in some order. Sequences may be finite or infinite.

A finite sequence is called a **tuple**. A sequence with **k** objects is called a **k-tuple**.

An **ordered pair** is a 2-tuple; that is, an ordered sequence of two elements. We write ordered pairs in parentheses, for example **(a, b)**, and we call **a** the first element and **b** the second element of the pair.

The **Cartesian product** or **cross product** of two sets A and B, written **A × B**, is the set of all ordered pairs wherein the first element is a member of A and the second element is a member of B.

Cartessian Product of Two set → Cross Product

Set A, B

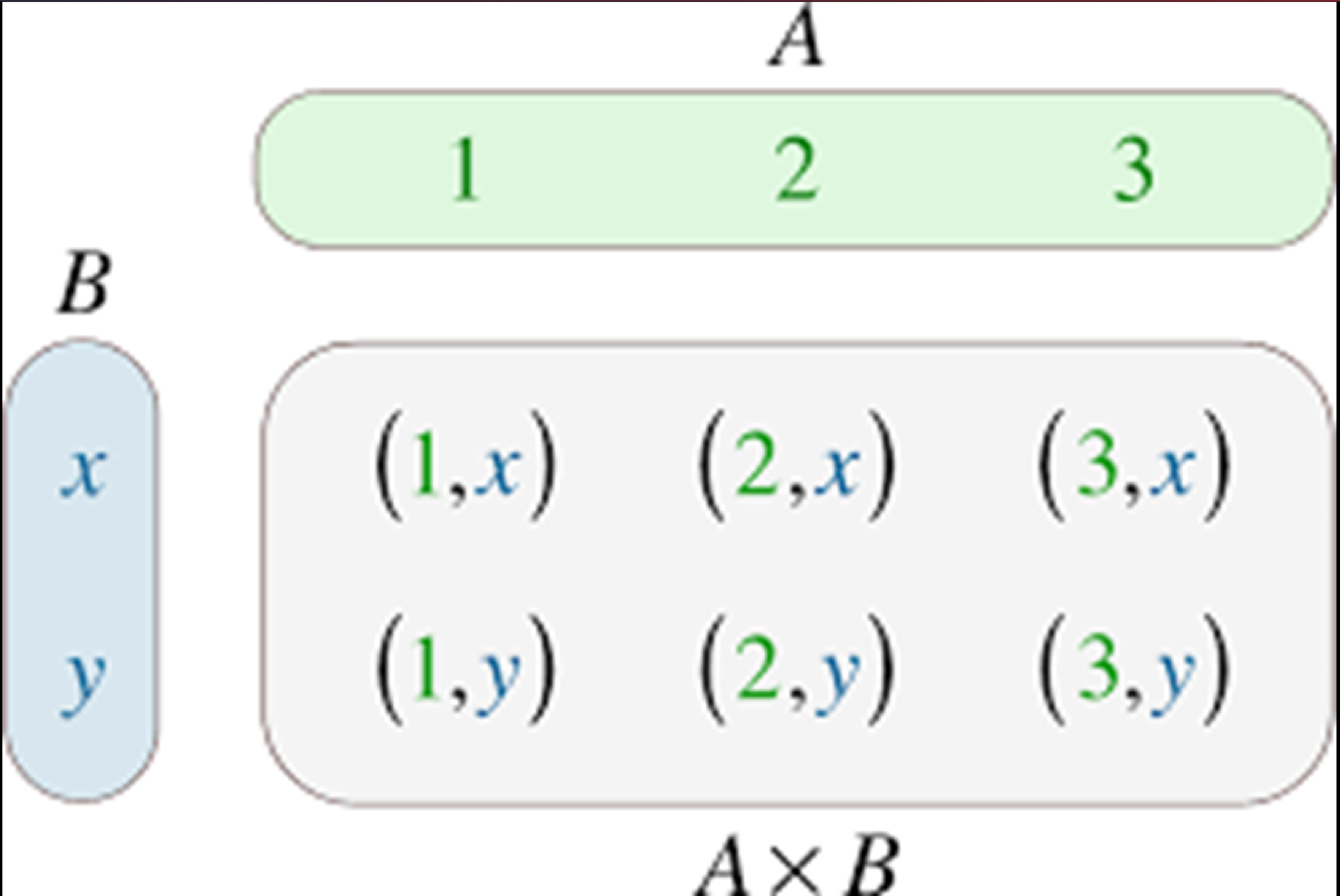
$A \times B = \{ (a, b) \mid a \in A \ \& \ b \in B \}$

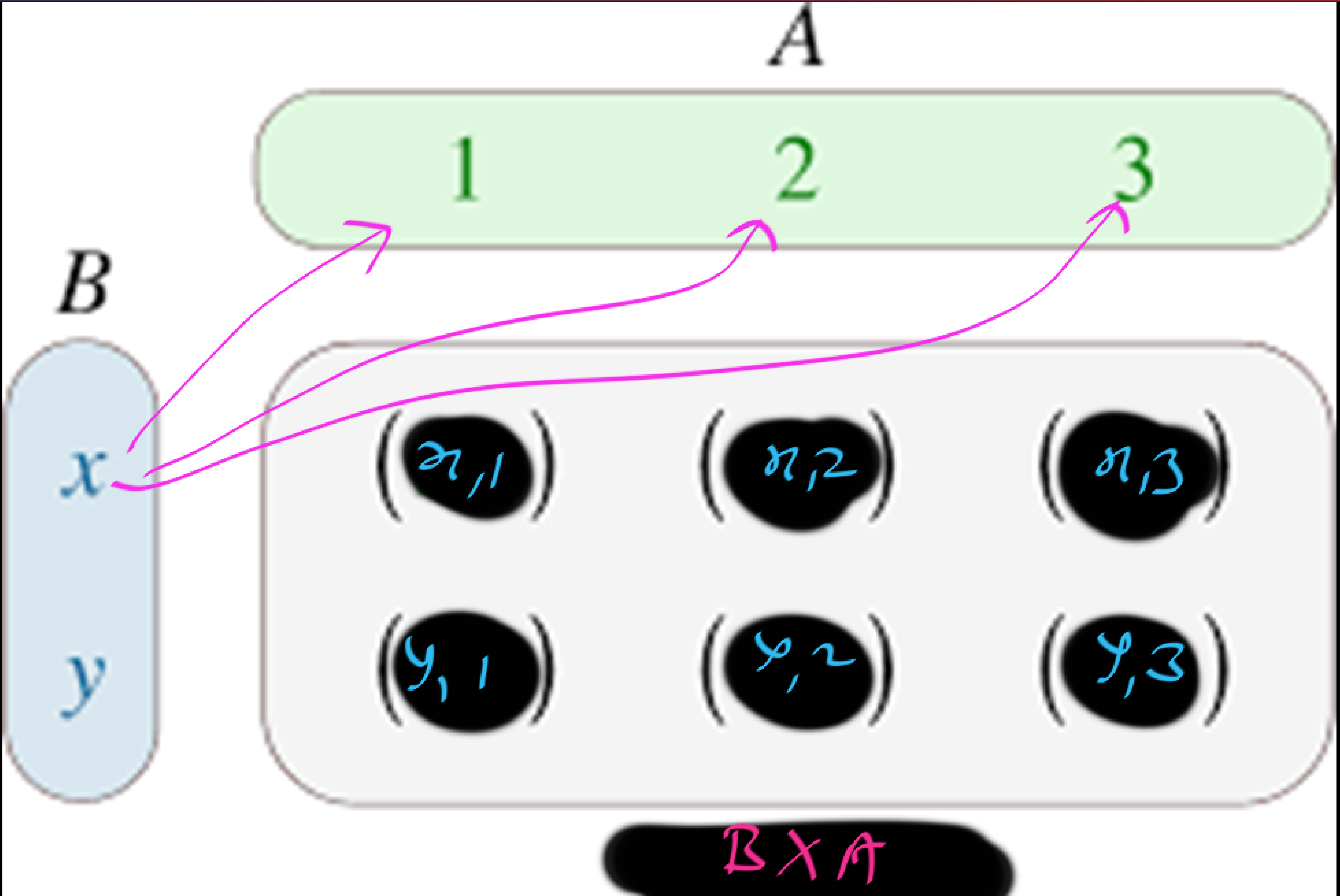
The Cartesian Product

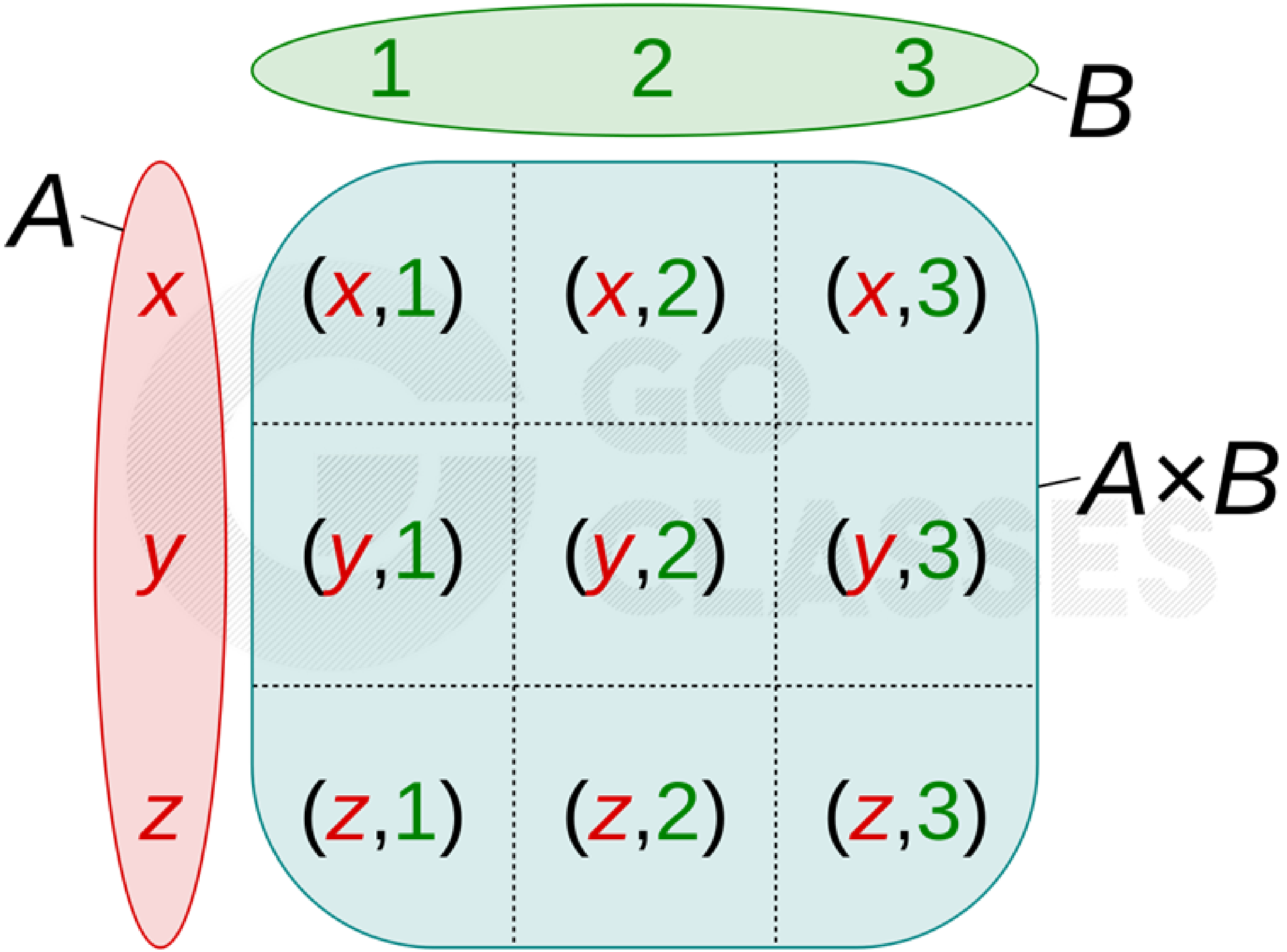
- Recall: The **power set** $\wp(S)$ of a set is the set of all its subsets.
- The **Cartesian Product** of $A \times B$ of two sets is defined as

$$A \times B \equiv \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

$$\underbrace{\{0, 1, 2\}}_{A} \times \underbrace{\{a, b, c\}}_{B} = \left\{ \begin{array}{l} (0, a),(0, b),(0, c), \\ (1, a),(1, b),(1, c), \\ (2, a),(2, b),(2, c) \end{array} \right\}$$


A
1 2 3
B
x
y
(1,x) (2,x) (3,x)
(1,y) (2,y) (3,y)
A × B


A
1
2
3
B
x
y
(x,1)
(x,2)
(x,3)
(y,1)
(y,2)
(y,3)
B X A


1
2
3
B
A
x
y
z
(x,1)
(x,2)
(x,3)
(y,1)
(y,2)
(y,3)
(z,1)
(z,2)
(z,3)
A×B

6.1 Cartesian Products

In the Cartesian plane (or x-y plane), we associate the set of points in the plane with the set of all ordered points (x, y), where x and y are both real numbers. The idea of a Cartesian product of sets replaces $\mathbb{R}$ in the description by some other set(s), and drops the geometric interpretation.

If A and B are sets, the *Cartesian product* of A and B is the set

$$A \times B = \{(a, b) : (a \in A) \text{ and } (b \in B)\}.$$

The following points are worth special attention:

- The Cartesian product of two sets is a set.
- The elements of that set are ordered pairs.
- In each ordered pair, the first component is an element of A, and the second component is an element of B.

Cartesian Products and Relations

Definition (Cartesian product) If A and B are sets, the *Cartesian product* of A and B is the set

$$A \times B = \{(a, b) : (a \in A) \text{ and } (b \in B)\}.$$

The following points are worth special attention: The Cartesian product of two sets is a set, and the elements of that set are ordered pairs. In each ordered pair, the first component is an element of A, and the second component is an element of B.

Example (Cartesian product) If $A = \{\{1, 2\}, \{3\}\}$ and $B = \{(a, b), (c, d)\}$, then

$$A \times B = \{(\{1, 2\}, (a, b)), (\{1, 2\}, (c, d)), (\{3\}, (a, b)), (\{3\}, (c, d))\}.$$

Determining $|A \times B|$. If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$ because there are $|A|$ choices for the first component of each ordered pair and, for each of these, $|B|$ choices for the second component of the ordered pair.

Cartesian product

Definition:

The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is defined as the set of ordered tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$. That is:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Example Cartesian products

Examples:

- $\{1,2\} \times \{3,4,5\} = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$.
- {Male, Female} × {Married, Single} × {Student, Faculty} = {(Male, Married, Student), (Male, Married, Faculty), (Male, Single, Student), (Male, Single, Faculty), (Female, Married, Student), (Female, Married, Faculty), (Female, Single, Student), (Female, Single, Faculty)}.
- $R \times R = \{(x,y) \mid x \in R, y \in R\}$ is the set of point coordinates in the 2D plane.

Note:

If $|A| = n$; $|B| = m$

$|A \times B| = nm = n * m$

$\{(x, y) \mid x \in A \text{ \& } y \in B\}$

x: n choices

y: m choices

Cardinality of Cartesian product

Fact:

In general, if A_i's are finite sets, we have:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

$|A| = m; \quad |B| = n; \quad |C| = p$

$$|A \times B \times C| = mnp$$

$$= |A| * |B| * |C|$$

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