



Lattice

Complete Summary



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GATE CSE AIR 53; AIR 67; AIR 107; AIR 206

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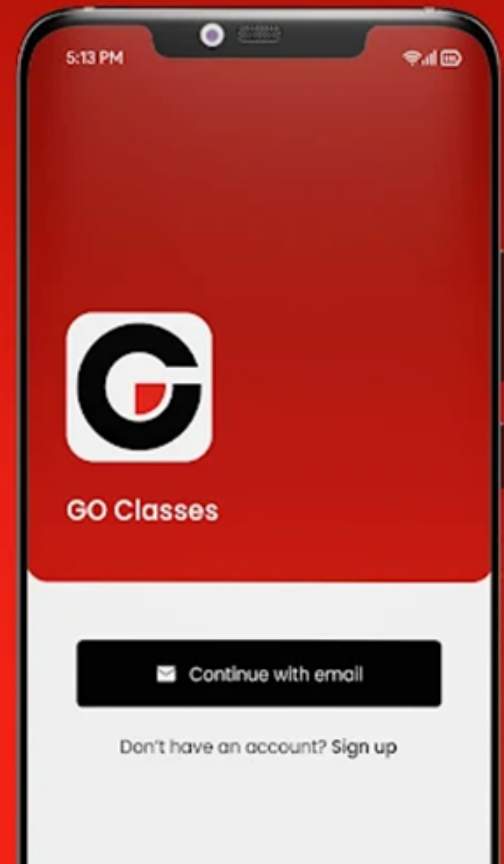
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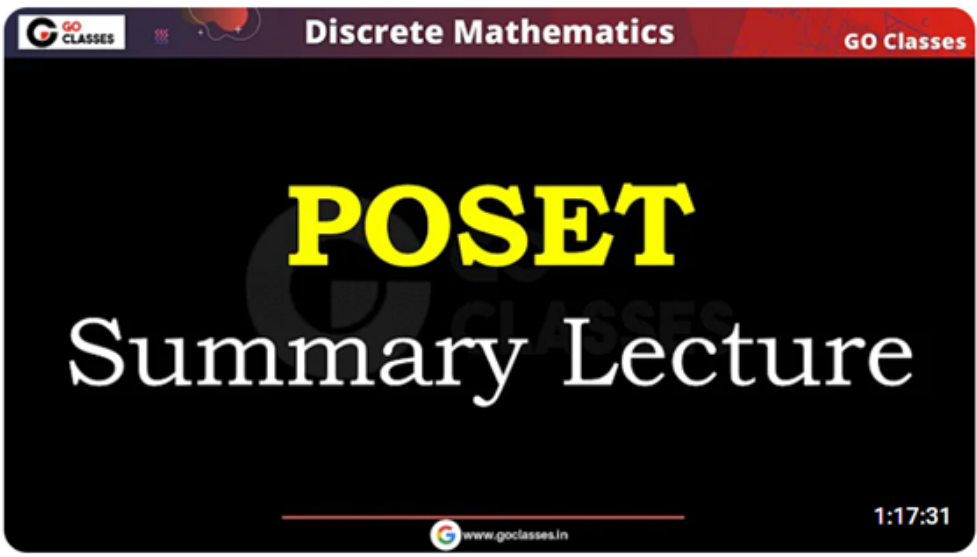
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Lattice

Complete Summary



Partial Order Relations - Summary - Part 1 | POSET | Set Theory | Discrete Mathematics | With NOTES

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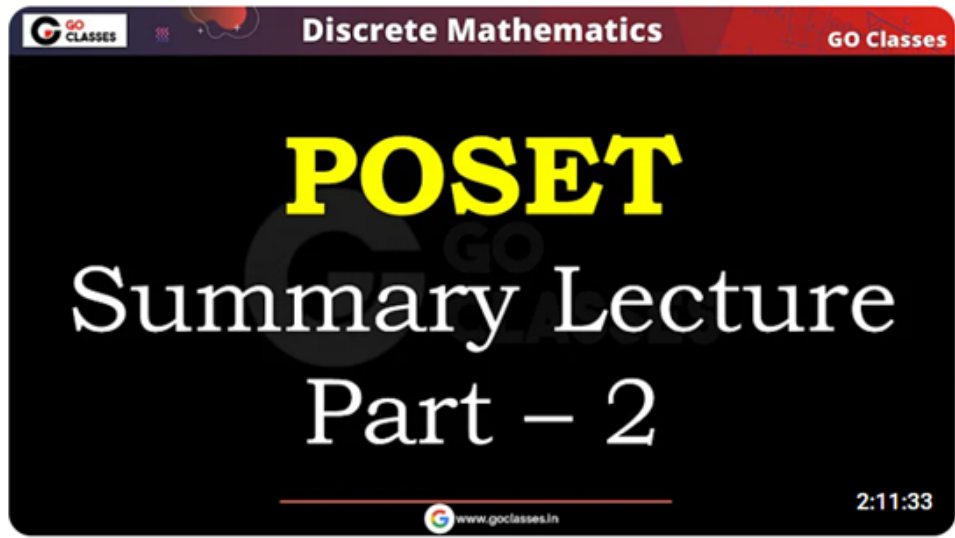
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go classes poset summary



Partial Order Relations - Summary - Part 2 | POSET | Set Theory | Discrete Mathematics | With NOTES

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In a POSET:

Upper Bound, Lower Bound

POSET (A, R) \rightarrow POR

Base set

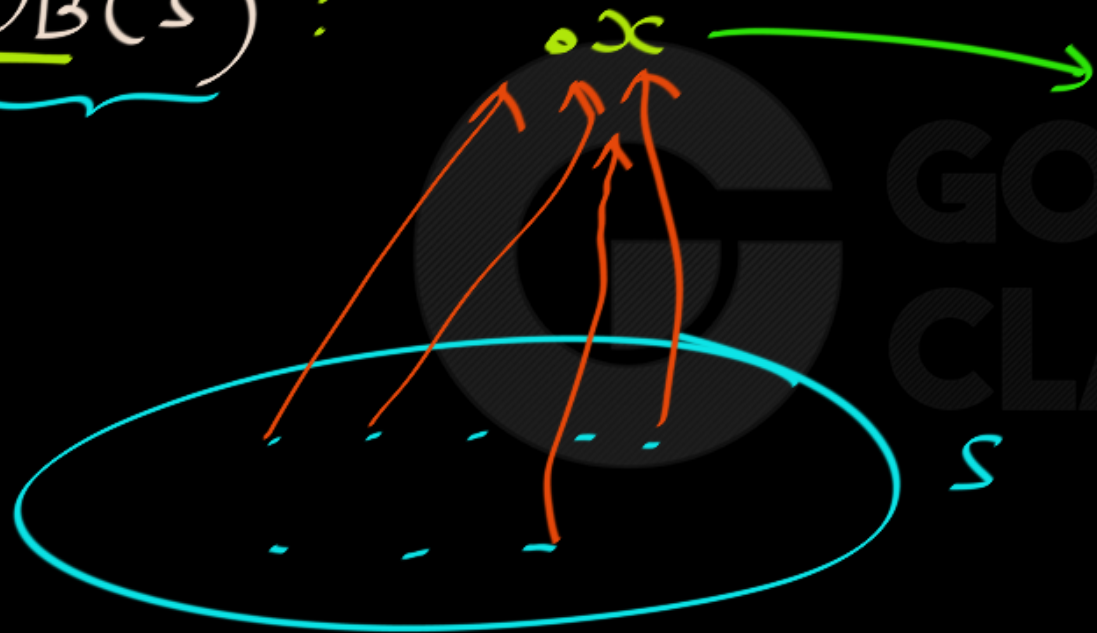
Let $S \subseteq A$

Upper bound of $S = UB(S)$

Lower bound of $S = LB(S)$

POSET (A, R) ; Let $S \subseteq A$

UB(S) :



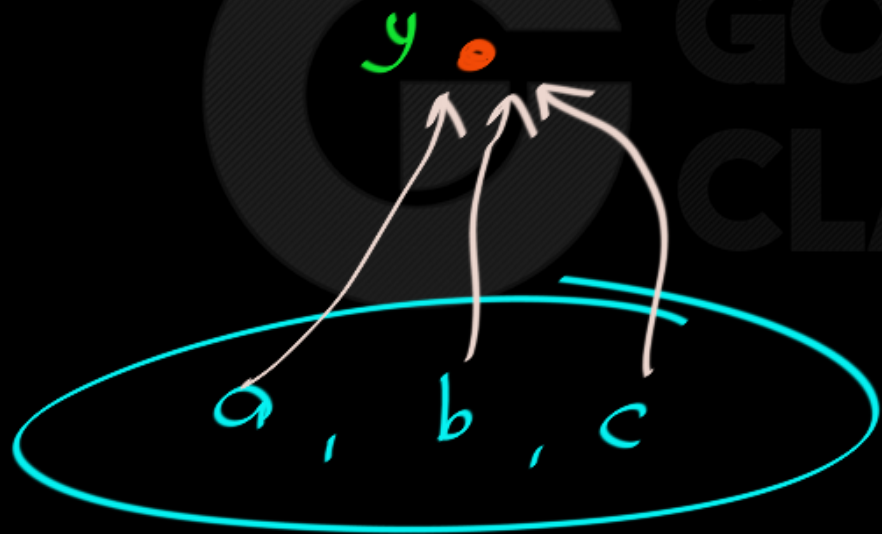
$x = UB(S)$ iff

$\forall b \in S$
 $b R x$

POSET (A, R) ; Let $S \subseteq A$

$$S = \{a, b, c\}$$

$$\underline{UB}(S) = ?$$

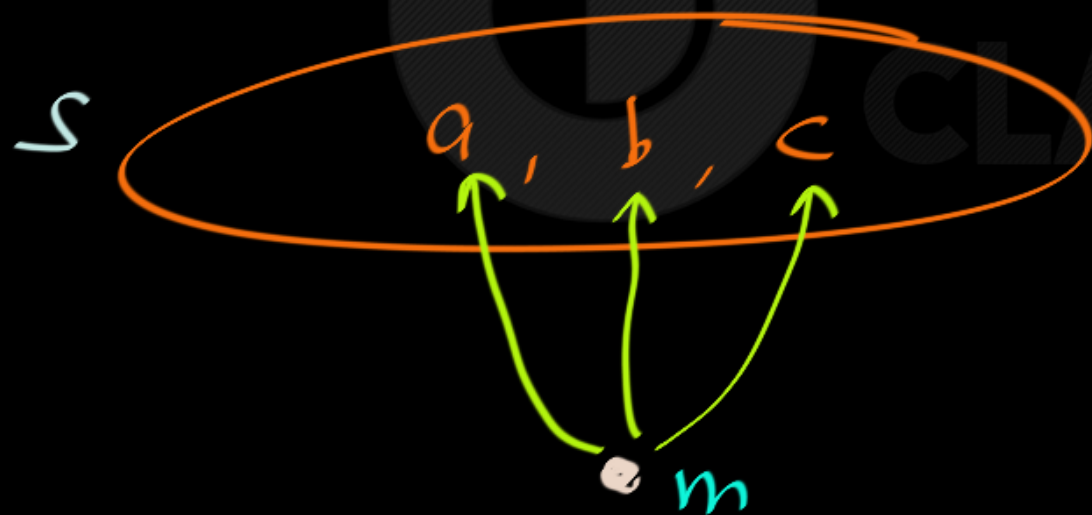


y is upper bound of S .

POSET (A, R) ; Let $S \subseteq A$

$S = \{a, b, c\}$

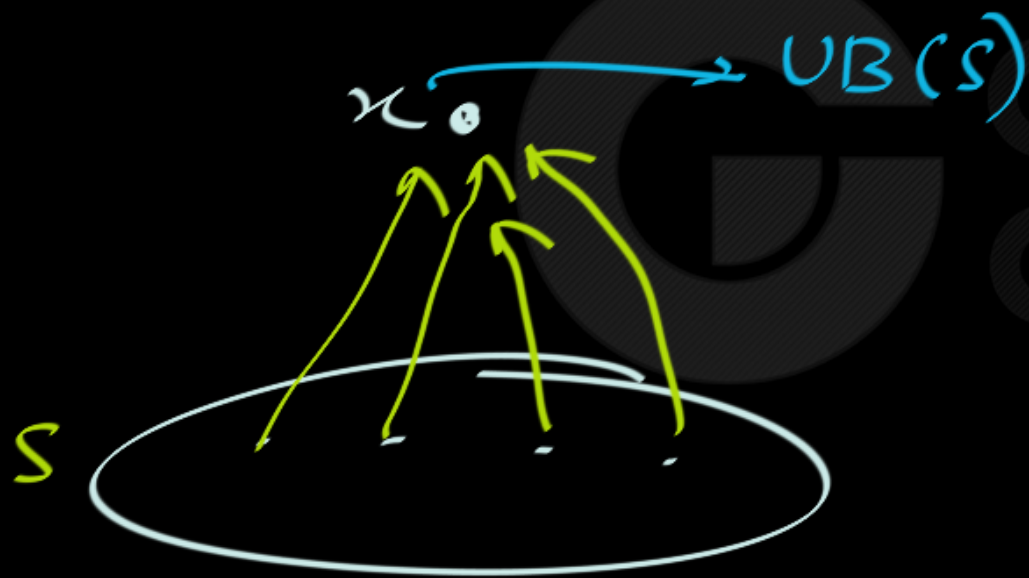
LB $(S) = ?$



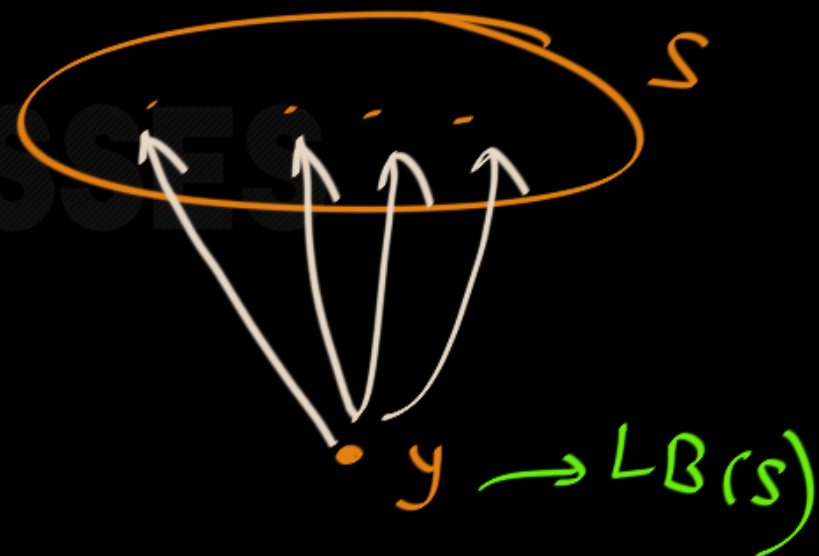
m is $LB(S)$ iff
 $\forall x \in S \quad m R x$

POSET (A, R) ; Let $S \subseteq A$

$UB(S)$



$LB(S)$



RDX



welcome

RDX is
UB { Uday, Manoj }





Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

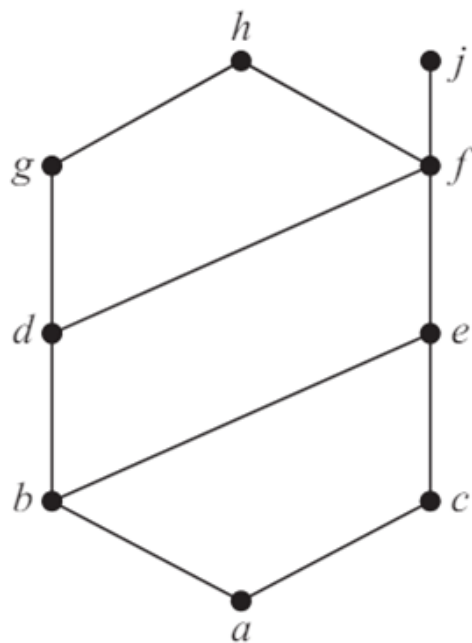


FIGURE 7 The Hasse Diagram of a Poset.

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

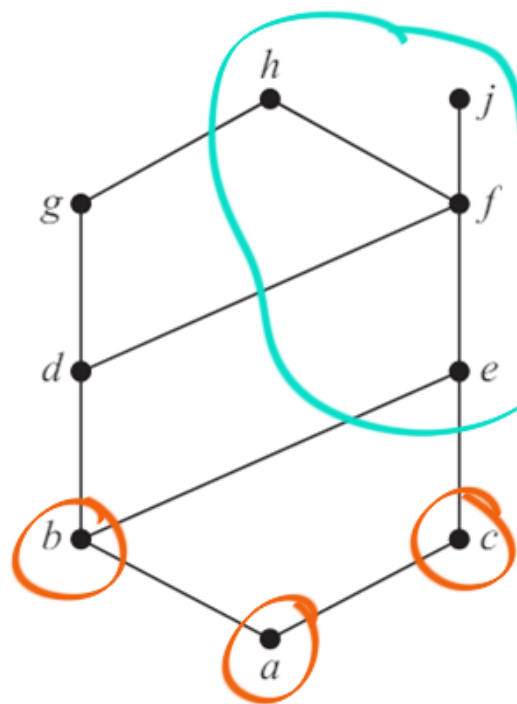
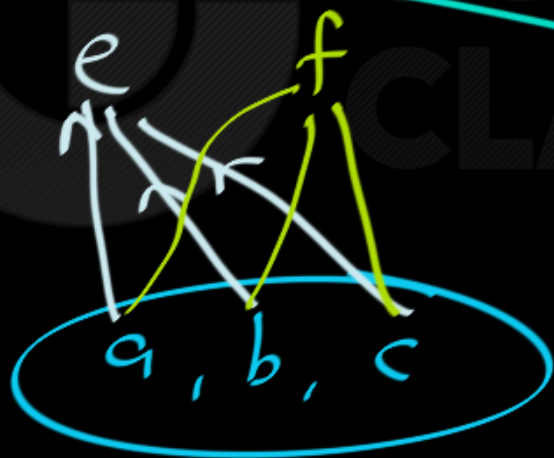


FIGURE 7 The Hasse Diagram of a Poset.

$$UB\{a, b, c\} = e, f, j, h$$



$$UB\{a, b, c\}$$

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

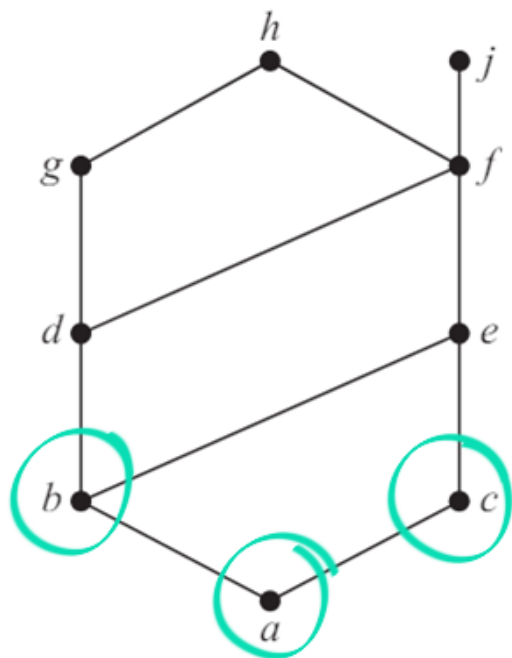


FIGURE 7 The Hasse Diagram of a Poset.

Handwritten notes on a black background:

$$LB \{a, b, c\} = a$$

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

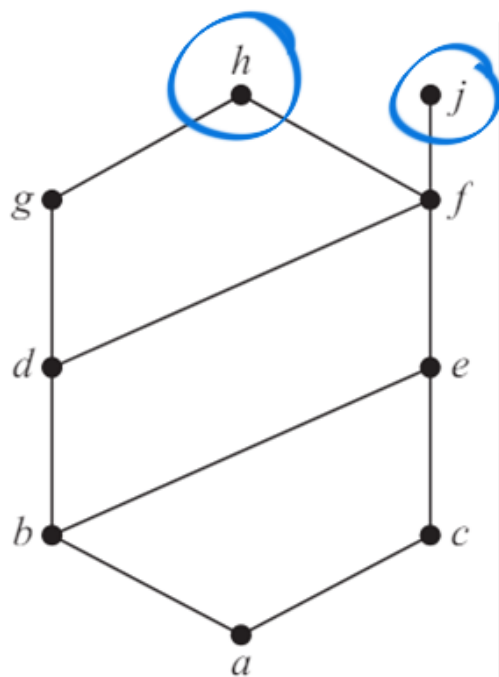
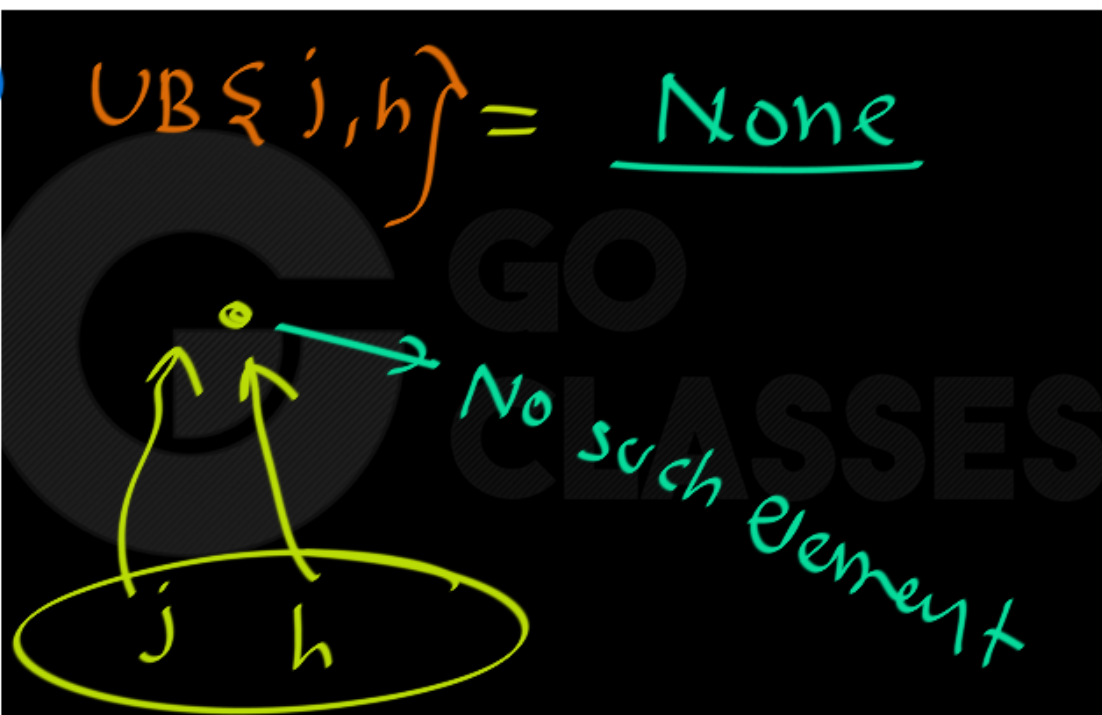


FIGURE 7 The Hasse Diagram of a Poset.



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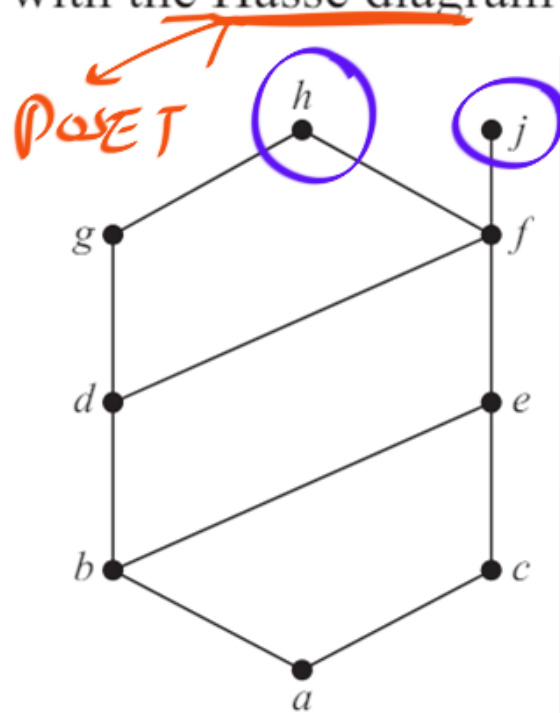
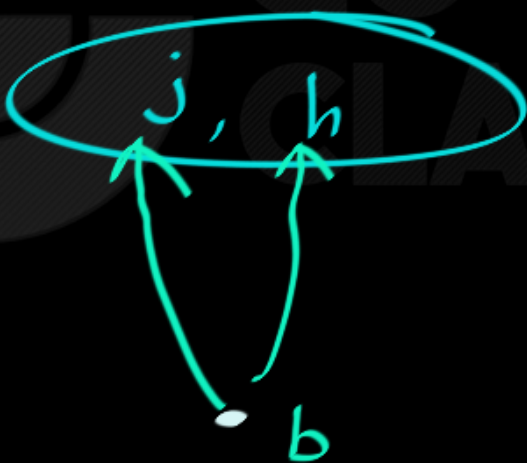


FIGURE 7 The Hasse Diagram of a Poset.

$$LB \{j, h\} = f, e, c, a, b, d$$



Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

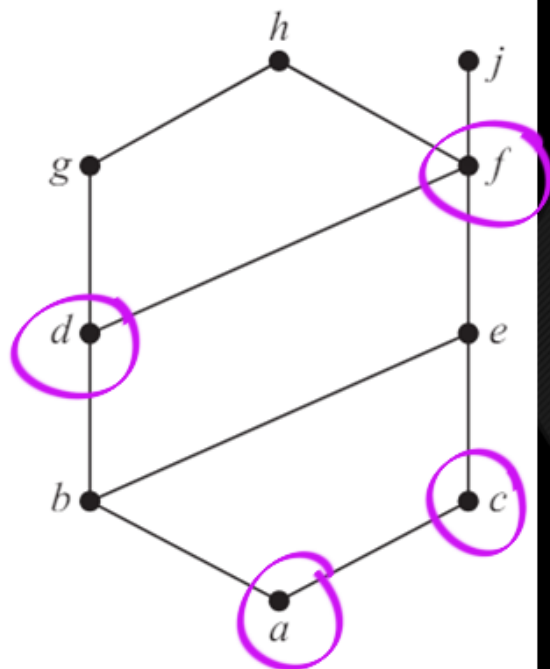


FIGURE 7 The Hasse Diagram of a Poset.

Handwritten notes on a black background:

$UB\{a, c, d, f\} = f, j, h$

$LB\{a, c, d, f\} = a$

Diagram illustrating the upper bounds of the subset $\{a, c, d, f\}$. The elements a, c, d, f are circled in orange. Lines connect them to their common upper bounds f, j, h .

Diagram illustrating the lower bounds of the subset $\{a, c, d, f\}$. The elements a, c, d, f are circled in green. Lines connect them to their common lower bound a .



Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

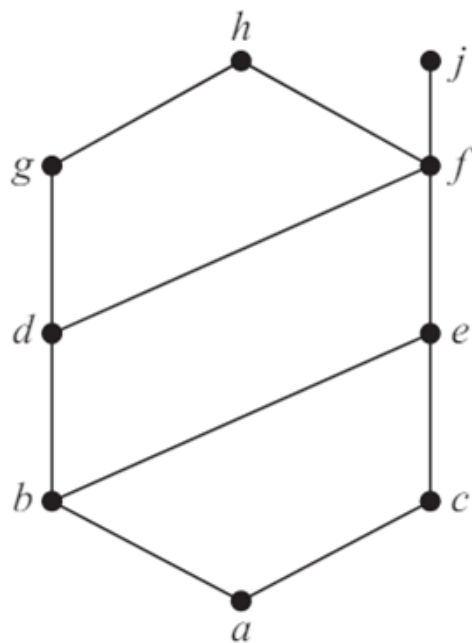


FIGURE 7 The Hasse Diagram of a Poset.

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j , and h , and its only lower bound is a . There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e , and f . The upper bounds of $\{a, c, d, f\}$ are f, h , and j , and its lower bound is a .

Upper and Lower Bounds

Definition: Let S be a subset of A in the poset (A, R) . If there exists an element a in A such that sRa for all s in S , then a is called an *upper bound*. Similarly for lower bounds.

in a subset A of a poset (S, \preceq) . If u is an element of S such that $a \preceq u$ for all elements $a \in A$, then u is called an **upper bound** of A . Likewise, there may be an element less than or equal to all the elements in A . If l is an element of S such that $l \preceq a$ for all elements $a \in A$, then l is called a **lower bound** of A .

An element x is an **upper bound** for a set A in a partially ordered set X if $x \succeq y$ for each $y \in A$. It is a **lower bound** if $y \succeq x$ for each $y \in A$.

4.3 Upper and lower bounds

4.16 Definition. For any subset X of a poset (P, \sqsubseteq) we define, for all $m \in P$:

- (a) m is an *upper bound* of X if: $(\forall x : x \in X : x \sqsubseteq m)$
- (b) m is a *lower bound* of X if: $(\forall x : x \in X : m \sqsubseteq x)$

□



In a POSET:

Least Upper Bound (LUB)

Greatest Lower Bound (GLB)

POSET (A, R) ; Let $S \subseteq A$

L UB $(S) =$ Least element in UB (S)

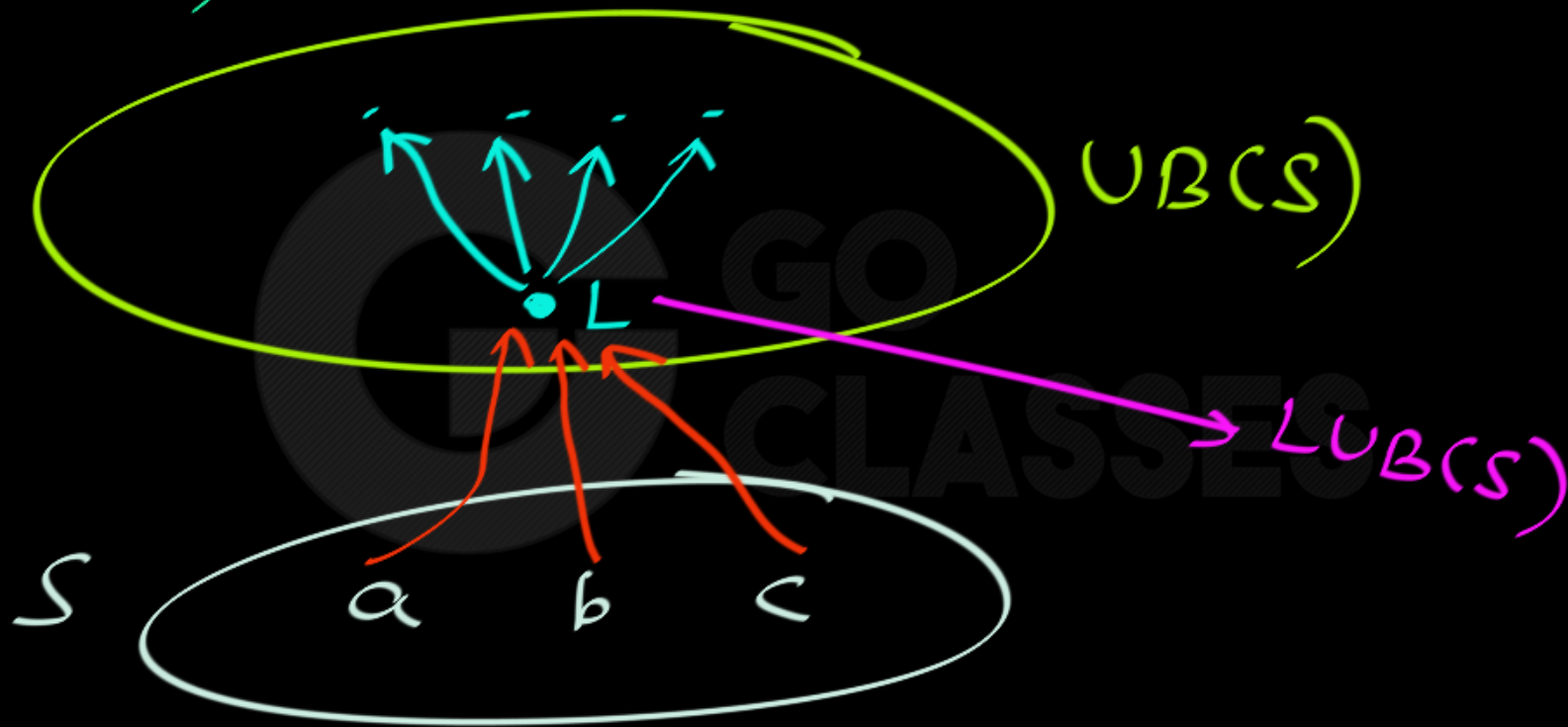
least

UB

Definition



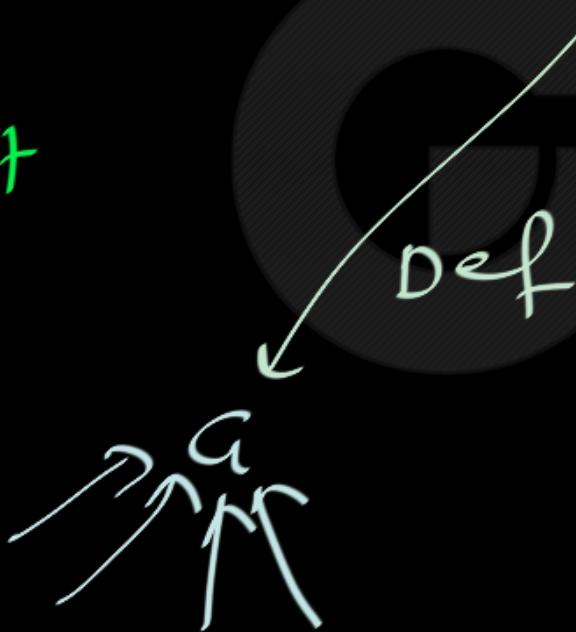
LUB(S)



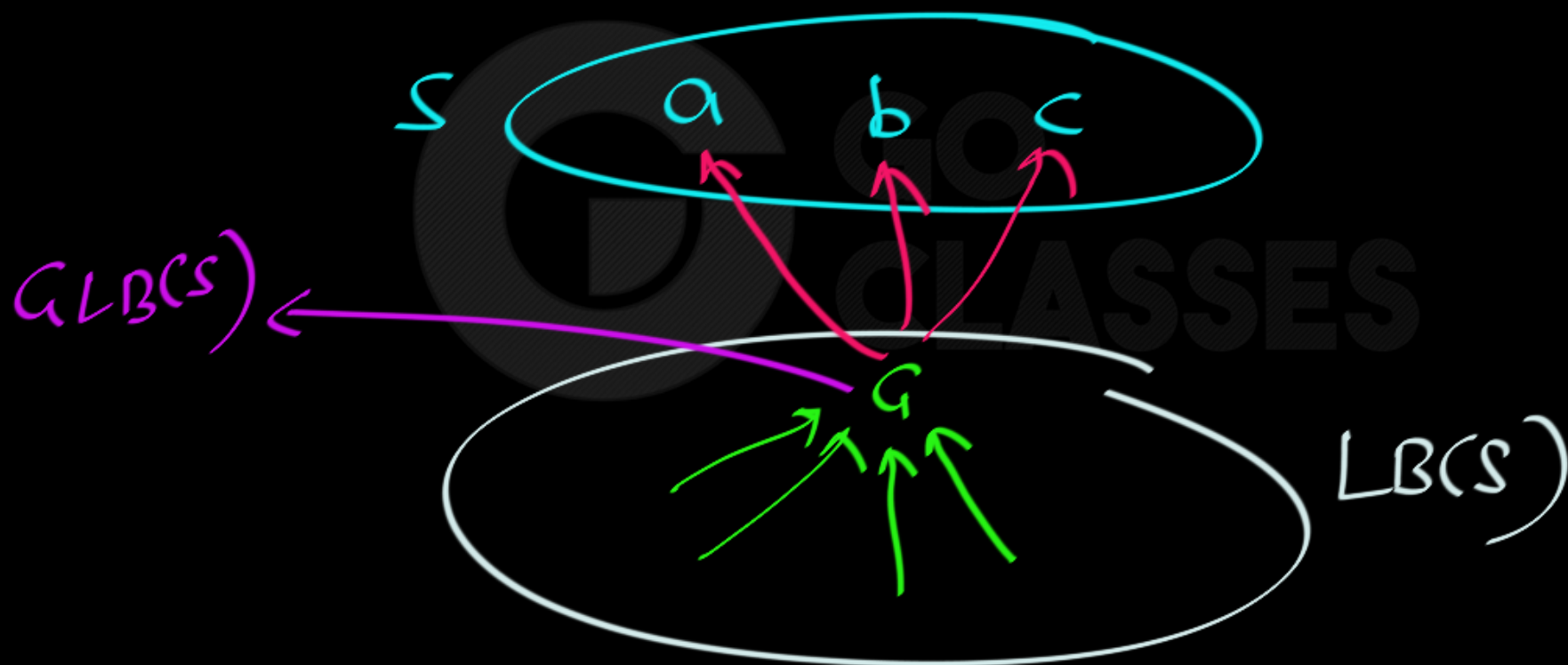
POSET (A, R) ; Let $S \subseteq A$

GLB(S) : Greatest element in LB(S)

Greatest



GLB(S)



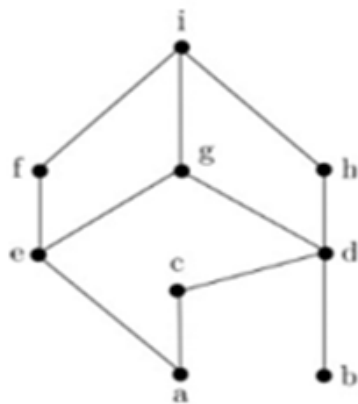


An element, $b \in X$, is the *greatest lower bound of A* iff the set of lower bounds of A is nonempty and if b is the greatest element of this set.

An element, $m \in X$, is the *least upper bound of A* iff the set of upper bounds of A is nonempty and if m is the least element of this set.



Consider the poset H represented by the following Hasse diagram.



Give each of the following. If any do not exist, explain why.

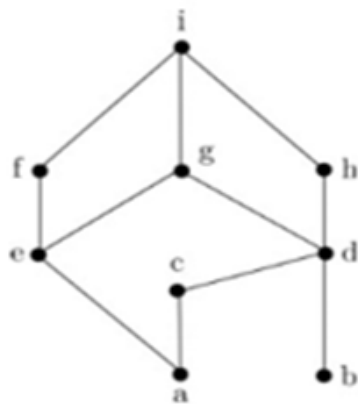
- (a) All minimal elements of H . (2)
- (b) The least element of H . (1)
- (c) All maximal elements of H . (2)
- (d) The greatest element of H . (1)
- (e) All lower bounds of $\{g, h\}$. (2)
- (f) The greatest lower bound of $\{g, h\}$. (1)
- (g) All upper bounds of $\{e, d\}$. (2)

- (h) The least upper bound of $\{e, d\}$. (1)
- (i) All lower bounds of $\{e, d\}$. (2)
- (j) Two elements which are not comparable. (2)



Consider the poset H represented by the following Hasse diagram.

minimal



Least



maximal



Greatest

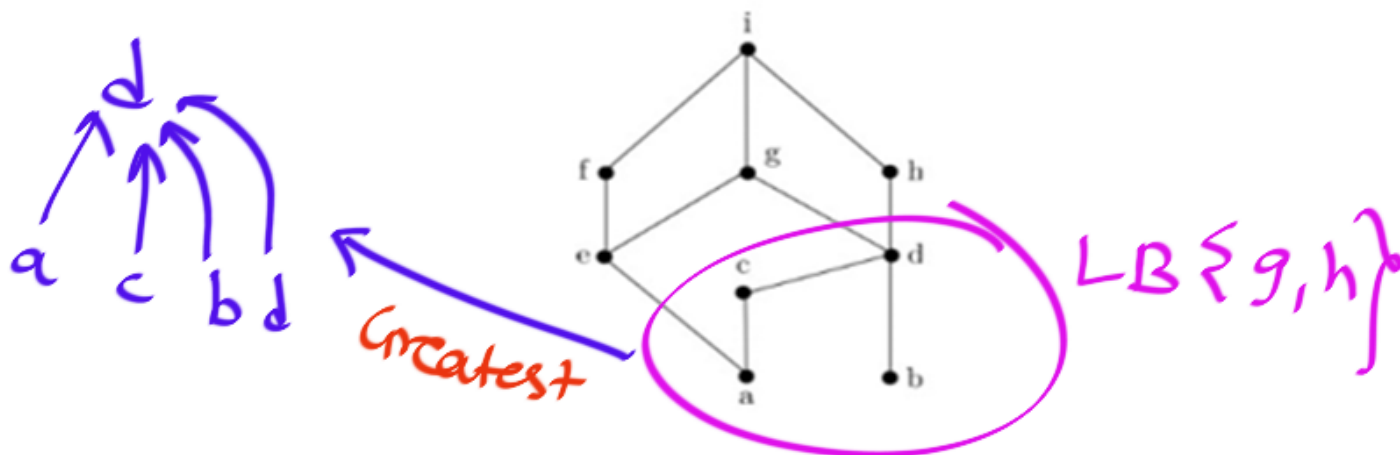


Give each of the following. If any do not exist, explain why.

- (a) All minimal elements of H . $\rightarrow a, b$ (2)
- (b) The least element of H . \rightarrow DNE (1)
- (c) All maximal elements of H . $\rightarrow i$ (2)
- (d) The greatest element of H . $\rightarrow i$ (1)
- (e) All lower bounds of $\{g, h\}$. $\rightarrow i$ (2)
- (f) The greatest lower bound of $\{g, h\}$. (1)
- (g) All upper bounds of $\{e, d\}$. (2)
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Consider the poset H represented by the following Hasse diagram.



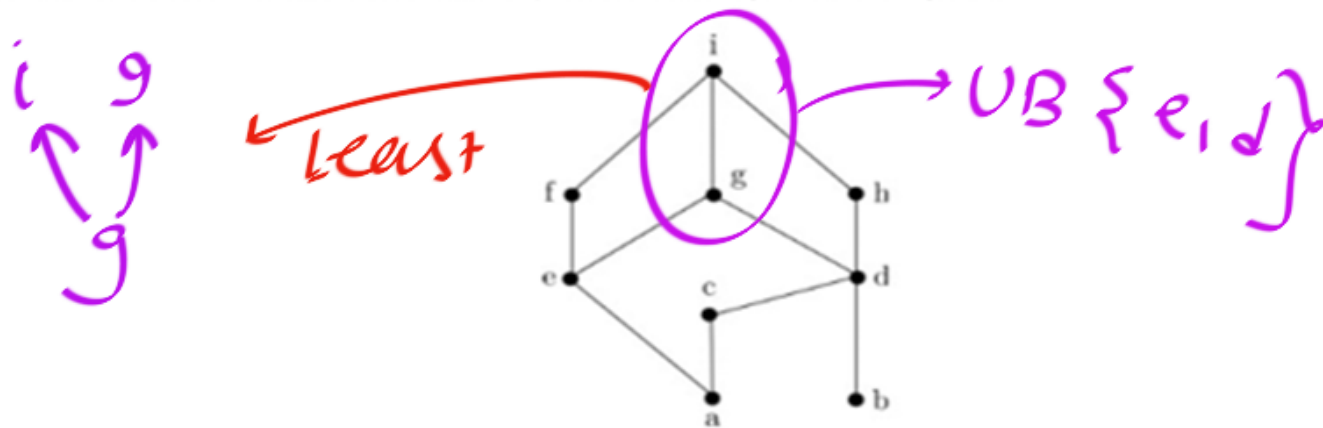
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Consider the poset H represented by the following Hasse diagram.

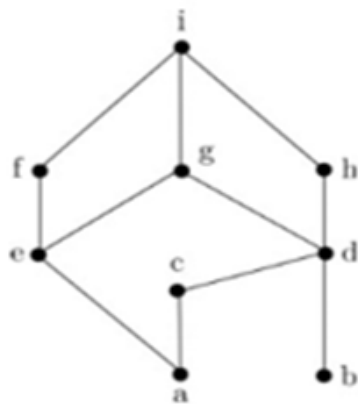


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- (j) Two elements which are not comparable. (2)



Consider the poset H represented by the following Hasse diagram.



Give each of the following. If any do not exist, explain why.

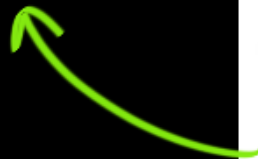
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- (g) All upper bounds of $\{e, d\}$. (2)
- (h) The least upper bound of $\{e, d\}$. (1)
- (i) All lower bounds of $\{e, d\}$. (2)
- (j) Two elements which are not comparable. (2)

$$LB\{e, d\} = \{a\}$$

$$GLB\{e, d\} = a$$

$$(a, b); (e, e)$$

g, h





13.1.1 Exercises for Section 13.1

A Exercises

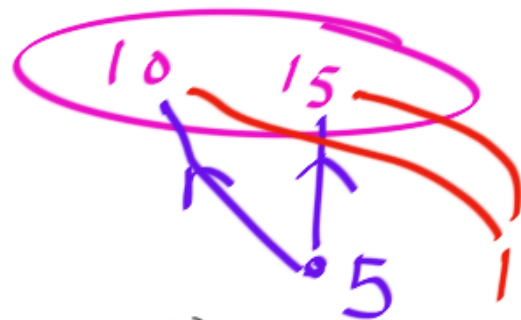
1. Consider the poset $(D_{30}, |)$, where $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.
 - (a) Find all lower bounds of 10 and 15.
 - (b) Find the greatest lower bound of 10 and 15.
 - (c) Find all upper bounds of 10 and 15.
 - (d) Determine the least upper bound of 10 and 15.

13.1.1 Exercises for Section 13.1

A Exercises

1. Consider the poset $(D_{30}, |)$, where $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

- Find all lower bounds of 10 and 15.
- Find the greatest lower bound of 10 and 15.
- Find all upper bounds of 10 and 15.
- Determine the least upper bound of 10 and 15.



Divides

Base set

$$LB \{10, 15\} = \{1, 5\}$$

$$GLB \{10, 15\} = 5 \checkmark$$



13.1.1 Exercises for Section 13.1

A Exercises

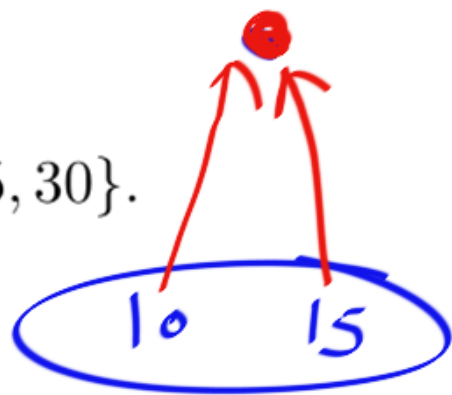
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(a) Find all lower bounds of 10 and 15.

(b) Find the greatest lower bound of 10 and 15.

(c) Find all upper bounds of 10 and 15.

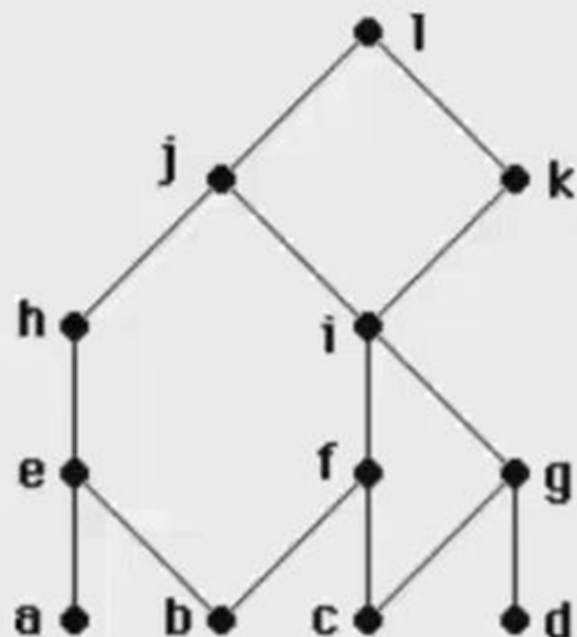
(d) Determine the least upper bound of 10 and 15.



$$\cup B \{10, 15\} = 30$$

$$= 30 \checkmark$$

5. Consider the poset with the following Hasse diagram.



HW

- Find all maximal elements of the poset.
- Find all minimal elements of the poset.
- Find the least element of the poset if it exists, or show that it does not exist.
- Find the greatest element of the poset if it exists, or show that it does not exist.
- What is the greatest lower bound of the set $\{a, b, c\}$?
- What is the least upper bound of the set $\{a, b, c\}$?



NOTE:
In ANY POSET,

Greatest Element

Least Element

LUB(S)

GLB(S)

→ Greatest

→ Unique if exists.

→ DNE
or

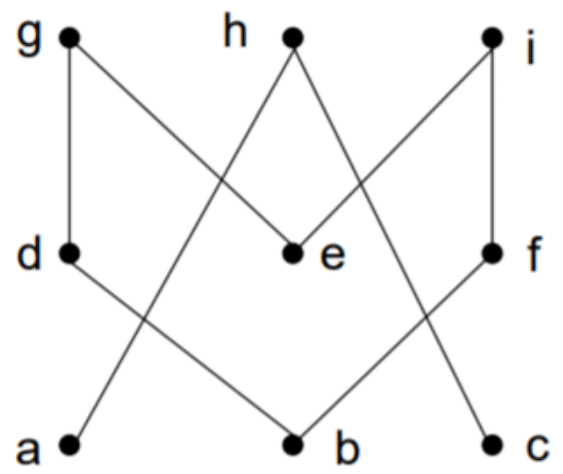
→ Unique



Extremal Elements:

Give lower/upper bounds & glb/lub of the sets:

$\{d,e,f\}$, $\{a,c\}$ and $\{b,d\}$

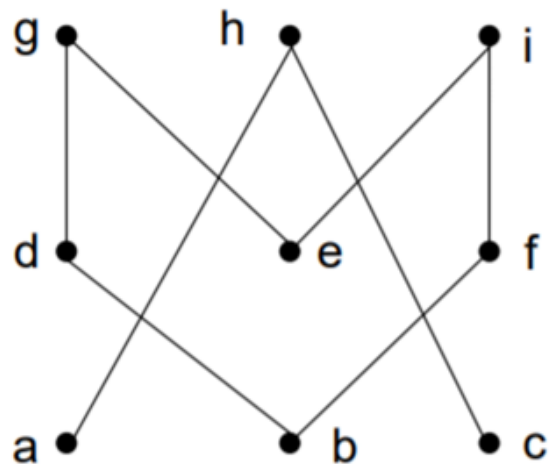




Extremal Elements:

Give lower/upper bounds & glb/lub of the sets:

$\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$



$$UB \{d, e, f\} = \text{None}$$

$$LUB \{d, e, f\} = \text{None}$$

$$LB \{d, e, f\} = \text{None}$$

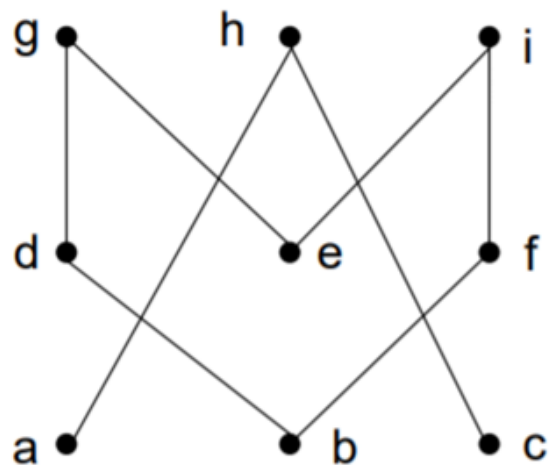
$$GLB \{d, e, f\} = \text{None}$$



Extremal Elements:

Give lower/upper bounds & glb/lub of the sets:

$\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$



$$UB \{a, c\} = h$$

$$LUB \{a, c\} = h$$

$$LB \{a, c\} = \text{None}$$

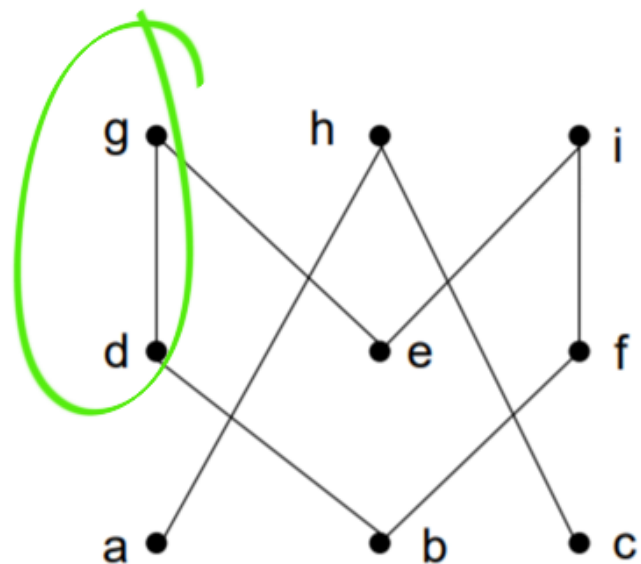
$$GLB \{a, c\} = \text{None}$$



Extremal Elements:

Give lower/upper bounds & glb/lub of the sets:

$\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$



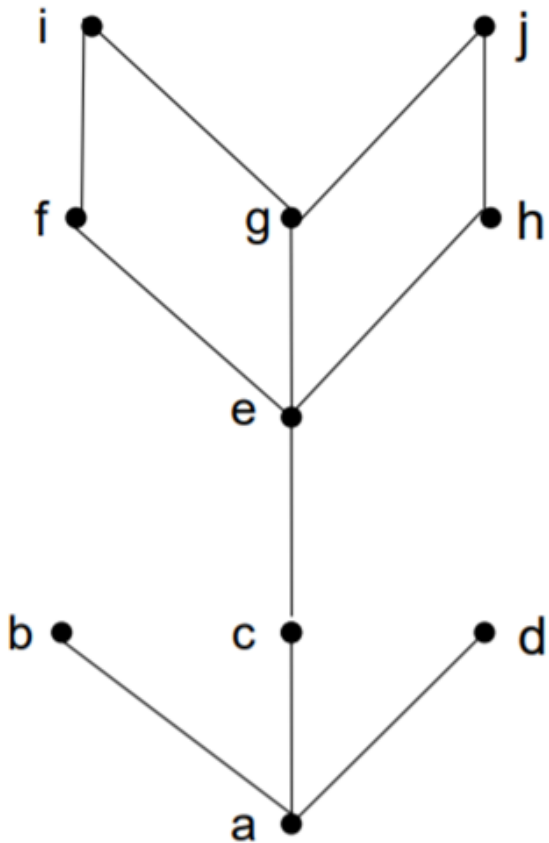
$$UB \{b, d\} = d, g$$

$$LUB \{b, d\} = d$$

$$LB \{b, d\} = b$$

$$GLB \{b, d\} = b$$

Extremal Elements: Example 3



- Minimal/Maximal elements?
 - Minimal & Minimum element: a
 - Maximal elements: b,d,i,j
- Bounds, glb, lub of $\{c,e\}$?
 - Lower bounds: $\{a,c\}$, thus glb is c
 - Upper bounds: $\{e,f,g,h,i,j\}$, thus lub is e
- Bounds, glb, lub of $\{b,i\}$?
 - Lower bounds: $\{a\}$, thus glb is c
 - Upper bounds: \emptyset , thus lub DNE

Definition

Let $\langle A, \leq \rangle$ be a poset and $S \subseteq A$.

1. $u \in A$ is an **upper bound** of S if $s \leq u$ for every $s \in S$.
2. $\ell \in A$ is a **lower bound** of S if $\ell \leq s$ for every $s \in S$.

Definition

Let $\langle A, \leq \rangle$ be a poset and $S \subseteq A$.

1. u is a **least upper bound** of S , denoted by $\underline{\text{LUB}}(S)$, if u is an upper bound of S and $u \leq u'$ for any upper bound u' of S .
2. ℓ is a **greatest lower bound** of S , denoted by $\underline{\text{GLB}}(S)$, if ℓ is a lower bound of S and $\ell' \leq \ell$ for any lower bound ℓ' of S .

Theorem

Any S has at most one LUB and at most one GLB.



Q:

For any POSET,

What is the Upper Bound, Lower Bound
of Entire POSET?

Q:

For any POSET,

What is the Upper Bound, Lower Bound
of Entire POSET?

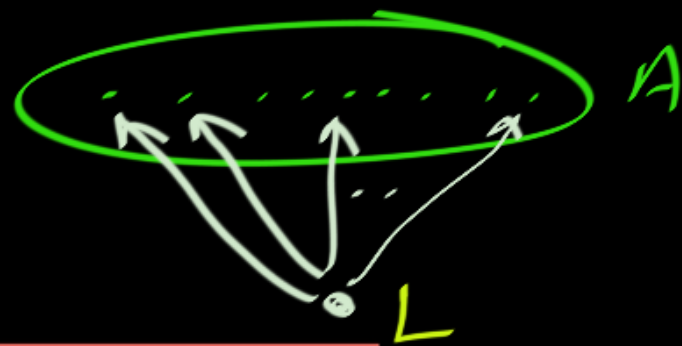
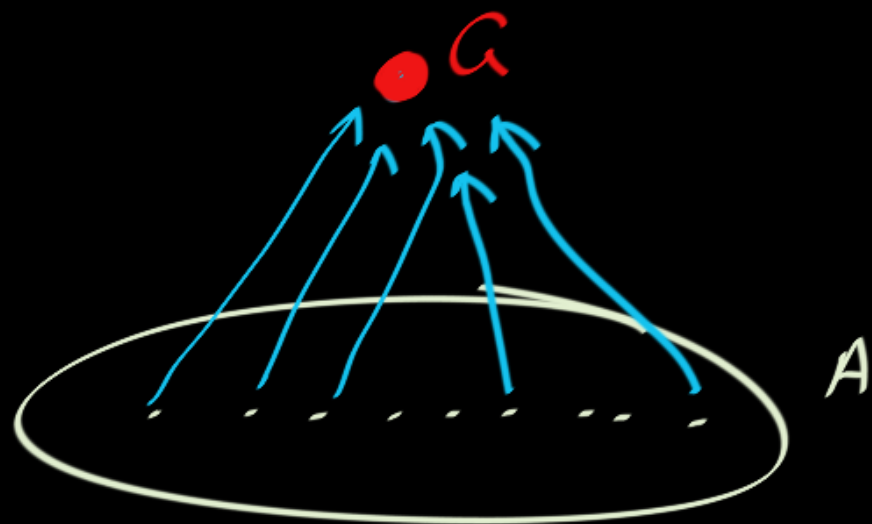
Greatest
element

Least
element

POSET (A, R)

UB(A) = Greatest Element

LB(A) = Least element





Important:

Quickly Finding GLB, LUB of Two Elements

In a POSET (X, R) ,

For any two elements a, b :

$$\underline{\text{LUB}}\{a, b\} = a \vee b = a \cup b = \underline{\text{supremum}}\{a, b\}$$

OR *union*
Join $\{a, b\}$

$$\text{GLB}\{a, b\} = a \wedge b = a \cap b = \text{infimum}\{a, b\}$$

and *meet $\{a, b\}$*

POSET (A, R)

$a, b \in A$

$$\underline{LUB} \{a, b\} = a \vee b = \text{Join} \{a, b\} = a \cup b = \text{Supremum} \{a, b\}$$

$$GLB \{a, b\} = a \wedge b = \text{meet} \{a, b\} = a \cap b = \text{infimum} \{a, b\}$$

for two elements a, b :

$$\textcircled{1} \quad \left. \begin{aligned} a \vee a &= a \\ a \wedge a &= a \end{aligned} \right\}$$

$\textcircled{2}$ When a, b are "Comparable":

Assume $a R b$:

$$a \vee b = b; \quad a \wedge b = a$$

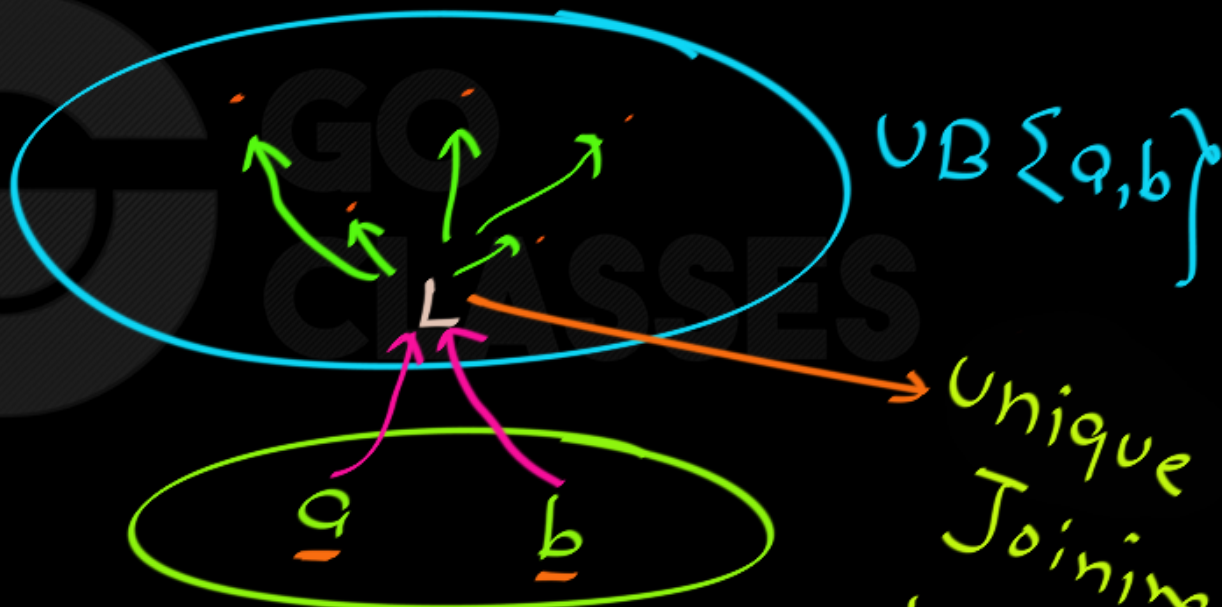
$$\begin{array}{l} b = a \vee b \\ | \\ a = a \wedge b \end{array}$$

for two elements a, b :

③ When a, b are NOT Comparable:

finding LUB:

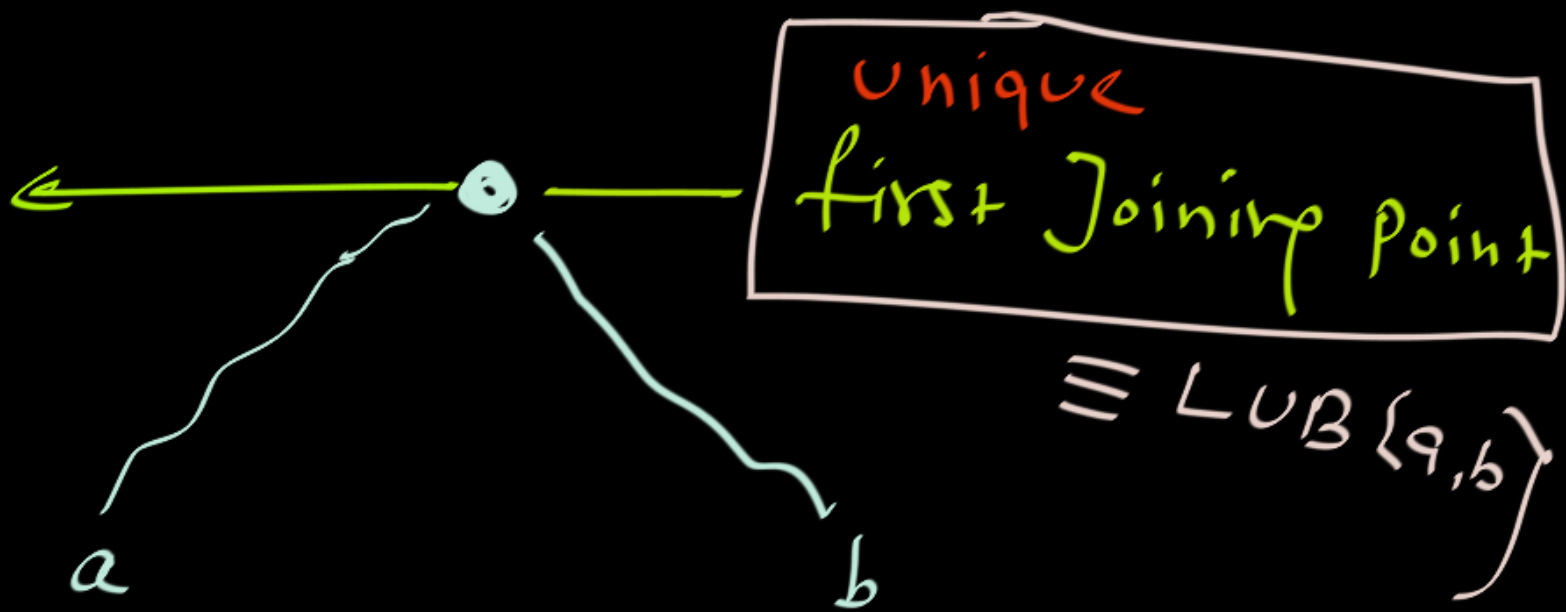
Least

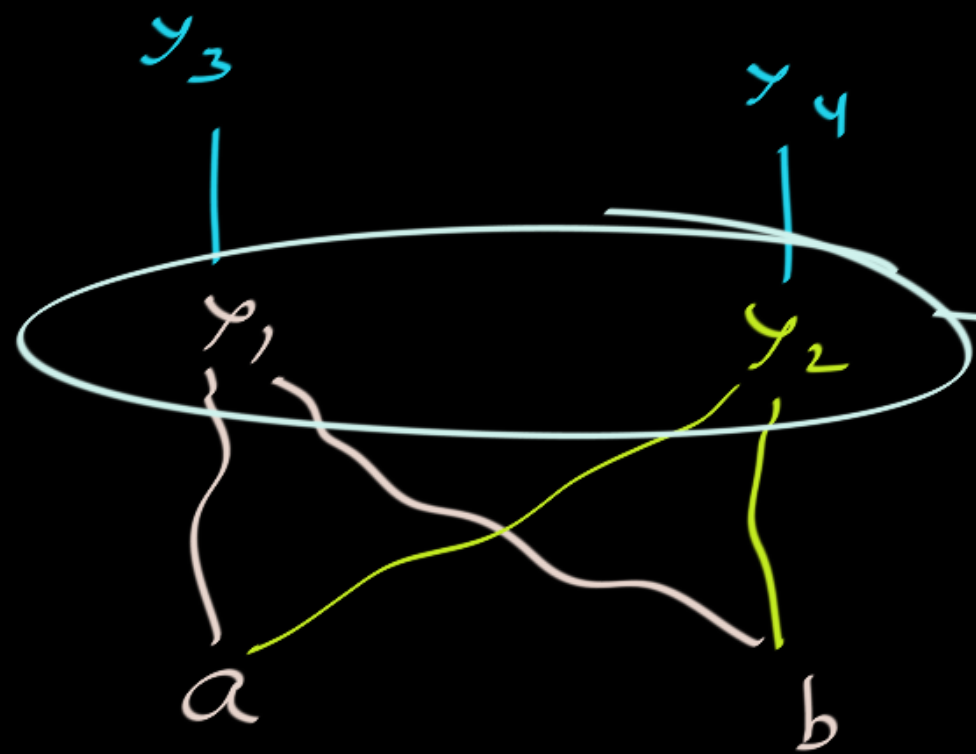


Unique first
Joining point
in up direction

LUB $\{a, b\}$

unique
first joining
point





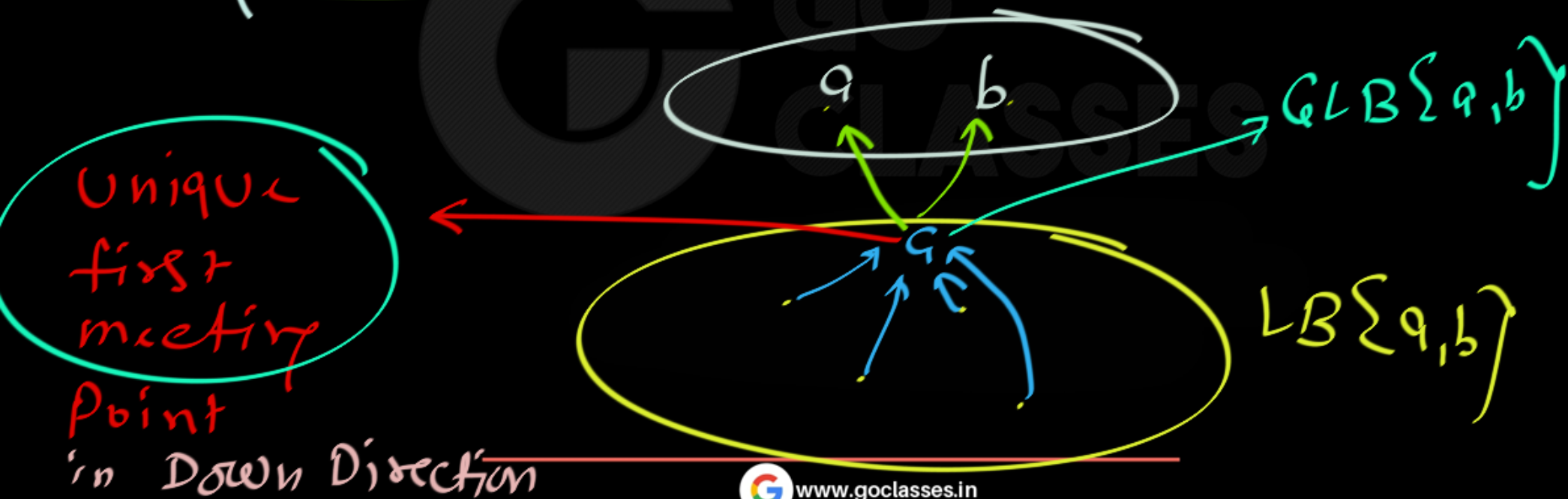
first Joining
Points but
Not unique

So; $LUB\{a, b\} = \underline{\underline{DNE}}$

for two elements a, b :

③ When a, b are NOT Comparable:

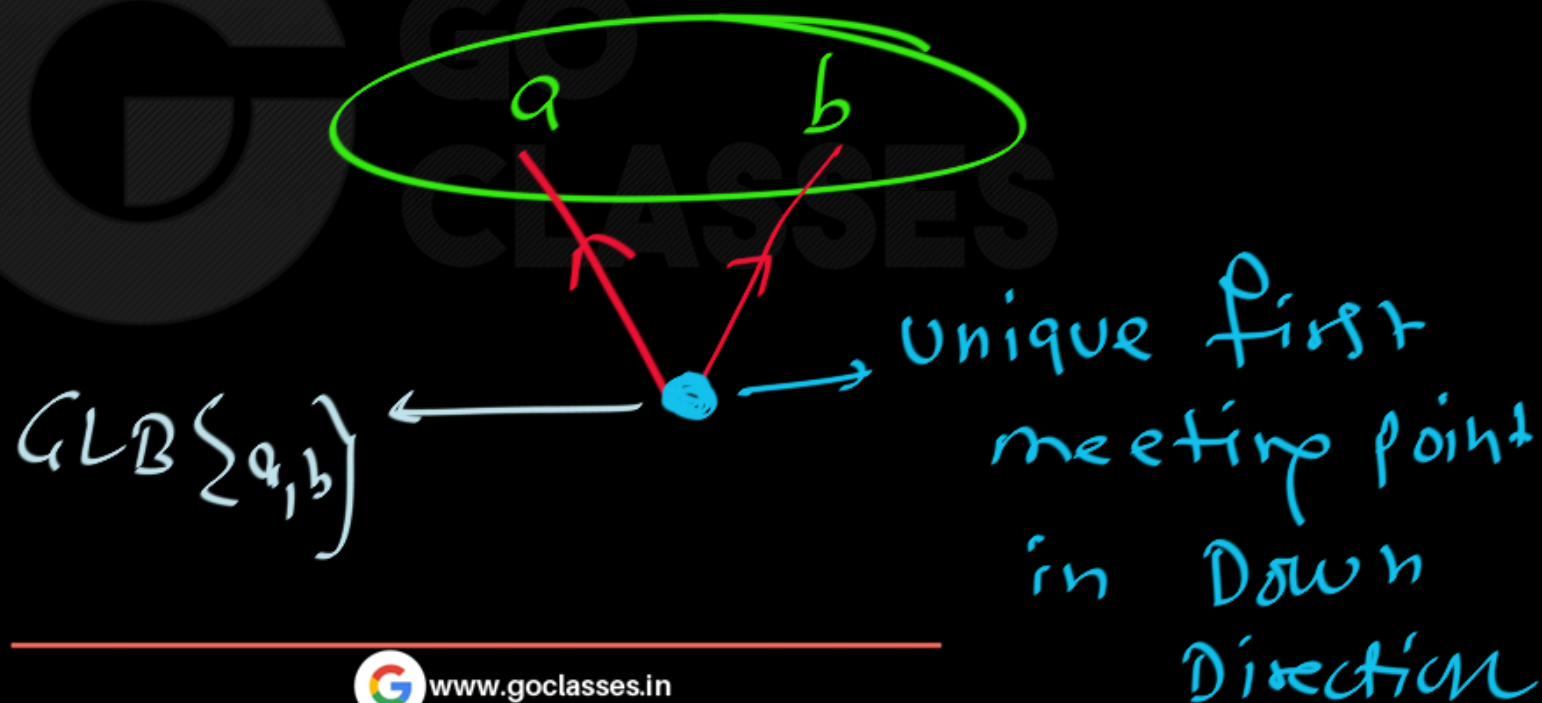
finding GLB



for two elements a, b :

③ When a, b are NOT Comparable:

$GLB\{a, b\}$





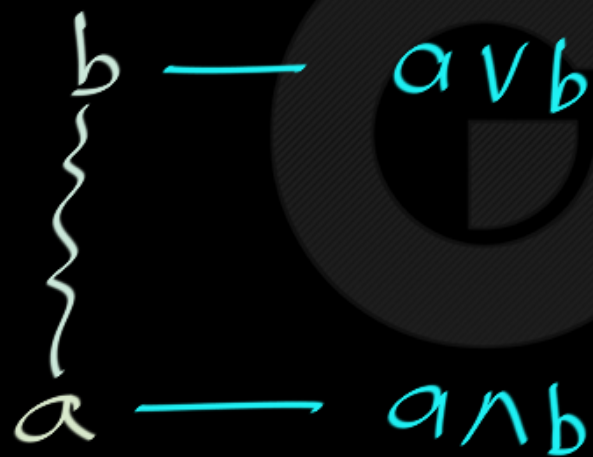
Finding LUB, GLB of One element:

$$\begin{cases} a \vee a = a \quad \checkmark \\ a \wedge a = a \quad \checkmark \end{cases}$$

Finding LUB, GLB of Two elements:

When elements are Comparable:

$$a R b$$



When $a R b$ then

$$a \vee b = b$$

$$a \wedge b = a$$



Finding LUB, GLB of Two elements:

When elements are Incomparable:

For LUB, unique first joining point in upper direction.

For GLB, unique first meeting point in down direction.

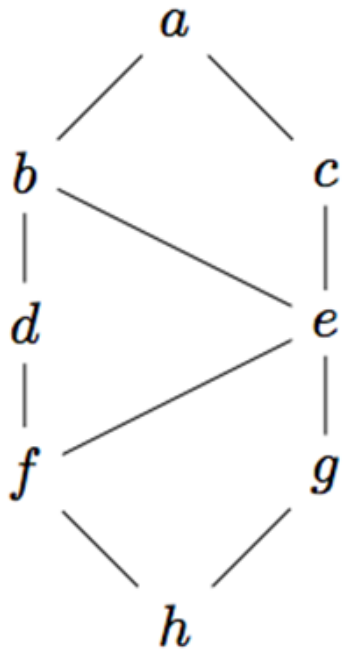
Finding LUB, GLB of more than Two elements:

For LUB, find UB then in that UB, find Least element.

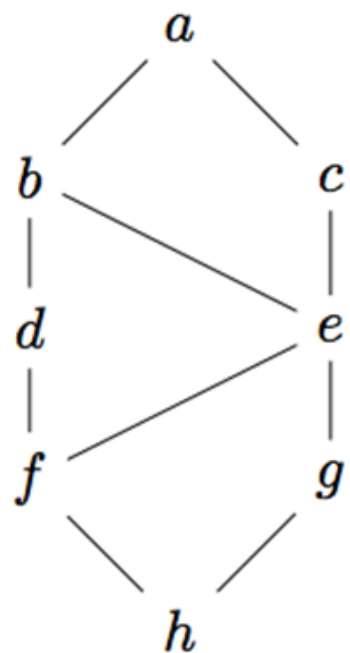
For GLB, find LB then in that LB, find Greatest element.



Q: Determine the least upper bound and greatest lower bound of all pairs of elements when they exist. Indicate those pairs that do not have a least upper bound (or a greatest lower bound).



Q: Determine the least upper bound and greatest lower bound of all pairs of elements when they exist. Indicate those pairs that do not have a least upper bound (or a greatest lower bound).



$$a \vee a = a$$

$$a \wedge a = a$$

$$f \wedge f = f$$

$$f \vee f = f$$

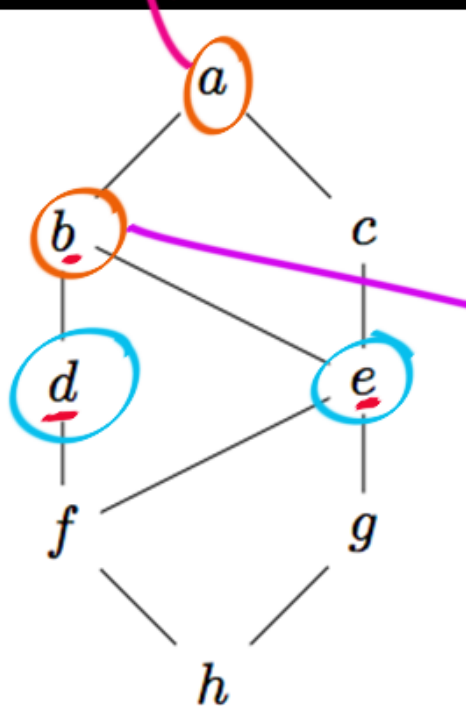
$$f \vee c = c$$

$$f \wedge c = f$$

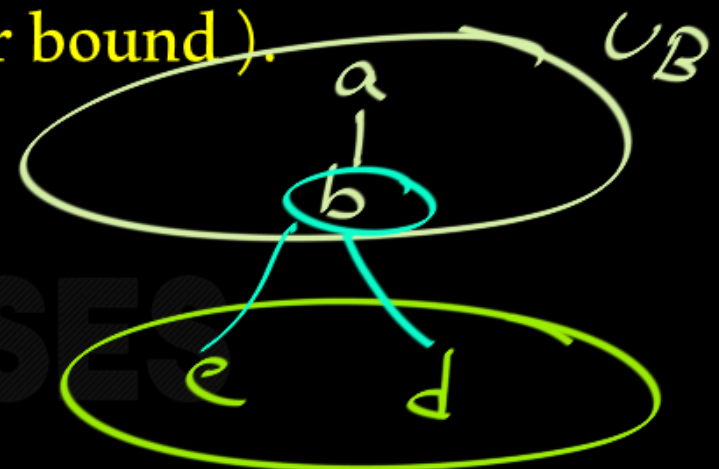
$$h \vee a = a$$

$$h \wedge a = h$$

Q: Determine the least upper bound and greatest lower bound of all pairs of elements when they exist. Indicate those pairs that do not have a least *second joining point* upper bound (or a greatest lower bound).

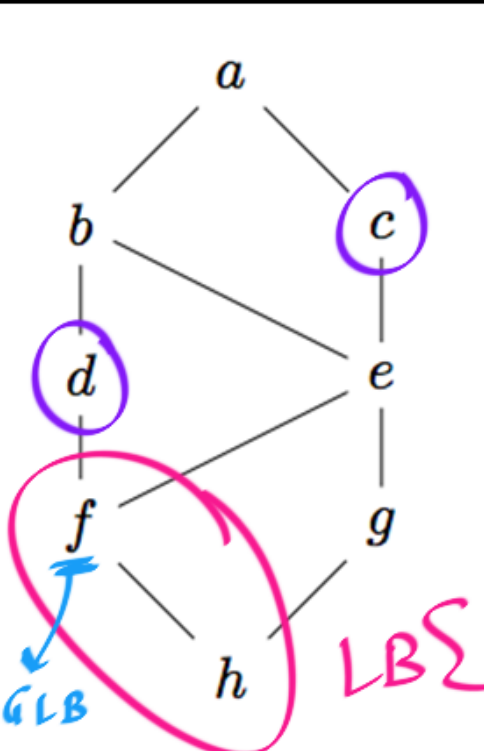


$$e \vee d = b$$



first joining point is unique

Q: Determine the least upper bound and greatest lower bound of all pairs of elements when they exist. Indicate those pairs that do not have a least upper bound (or a greatest lower bound).

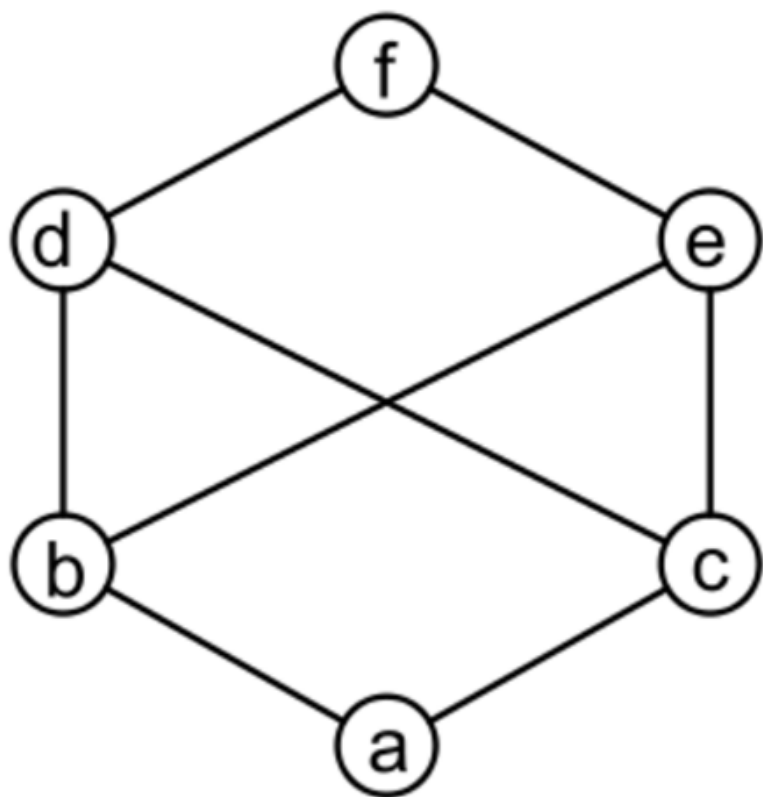


$$e \vee d = b ; e \wedge d = f$$

$$f \vee g = e ; f \wedge g = h$$

$$d \vee c = a ; d \wedge c = f$$

LBS {d, c}



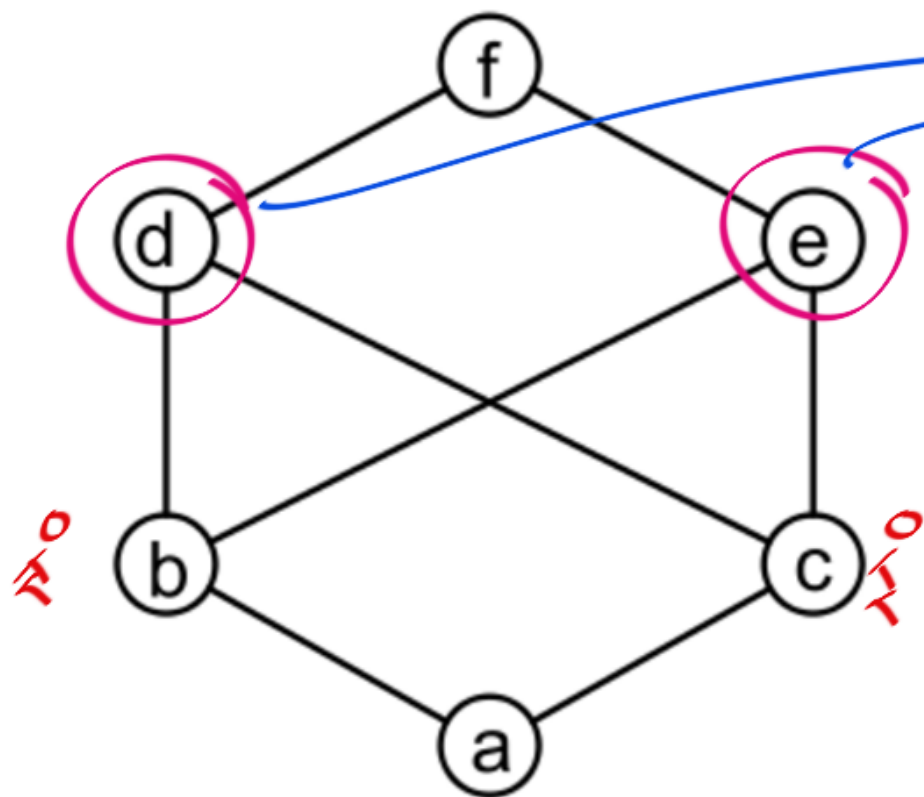
$$b \vee e = e$$

$$b \wedge e = b$$

$$d \wedge d = d$$

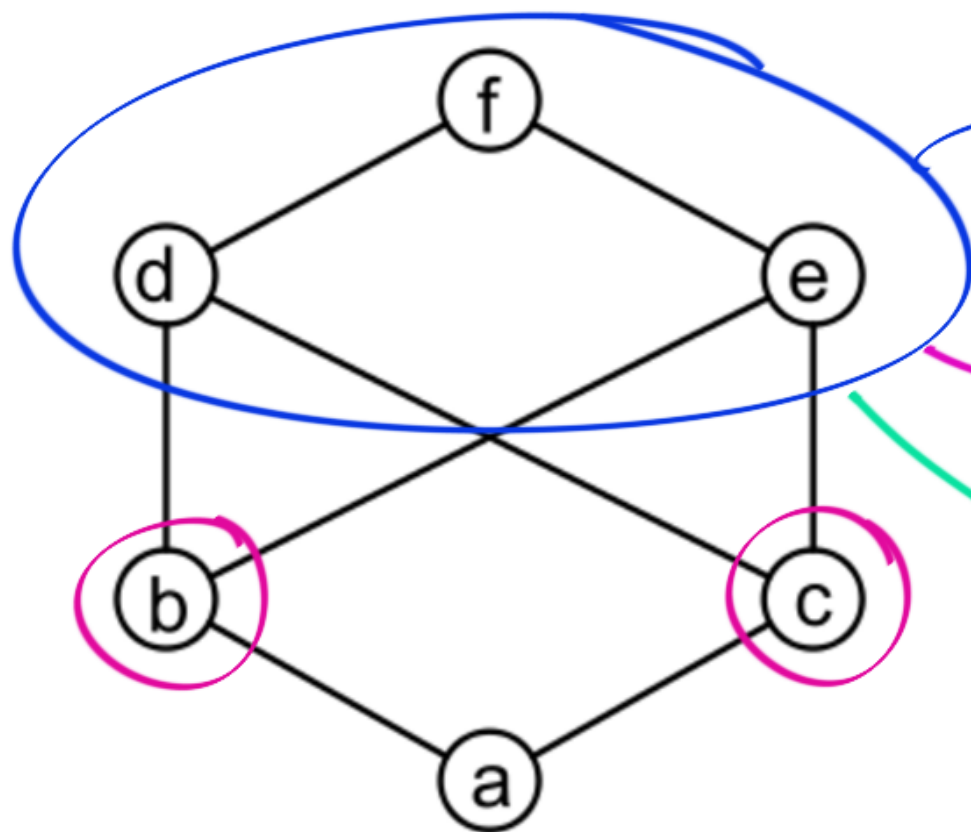
$$d \vee d = d$$

$$b \vee c = ?$$



first joining
points but
not unique.

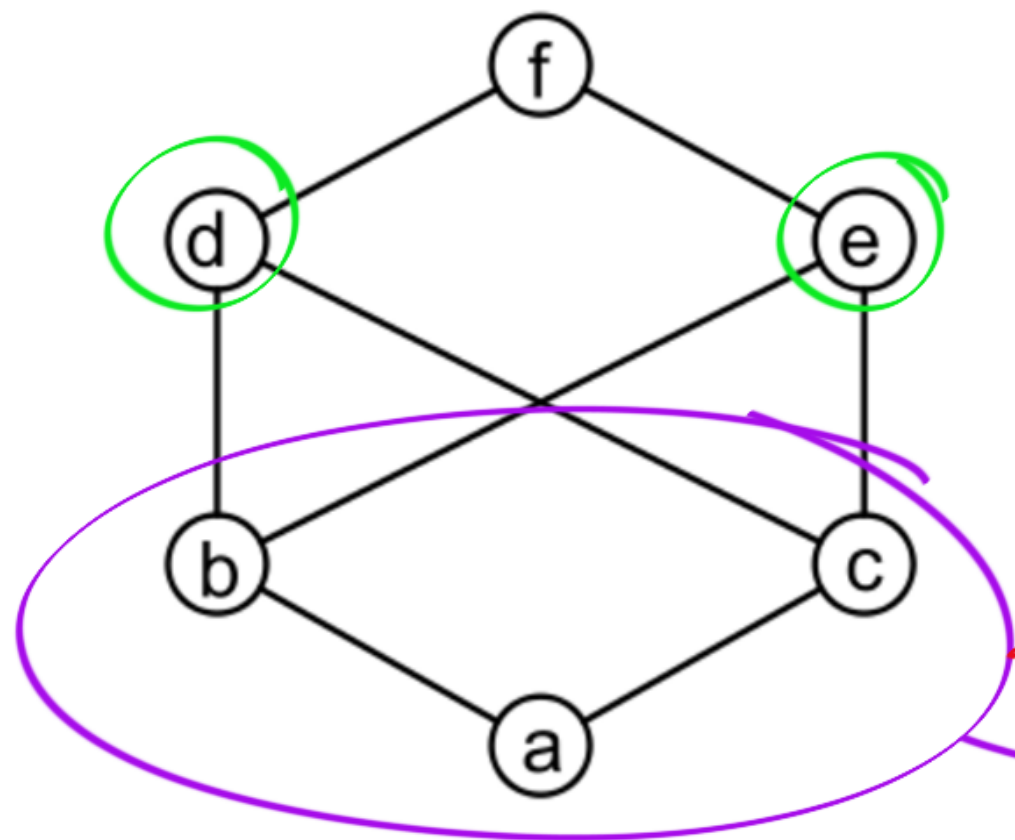
$b \vee c = \text{DNE}$



$$\cup B \{b, c\}$$

Least element
= DNE

$$\underline{b \vee c = \text{DNE}}$$



$$b \vee d = d$$

$$b \wedge d = b$$

$$d \wedge e = \text{DNE}$$

$$\text{GLB}\{d, e\} = \text{DNE}$$

→ No Greatest

→ $\text{LB}\{d, e\}$



+

+

Lattice



Definition

A **lattice** is a poset $\langle A, \leq \rangle$ in which any two elements a, b have an $\text{LUB}(a, b)$ and a $\text{GLB}(a, b)$.

We write $a \cup b = \text{LUB}(a, b)$ and $a \cap b = \text{GLB}(a, b)$. We also call them **join** and **meet**, respectively.

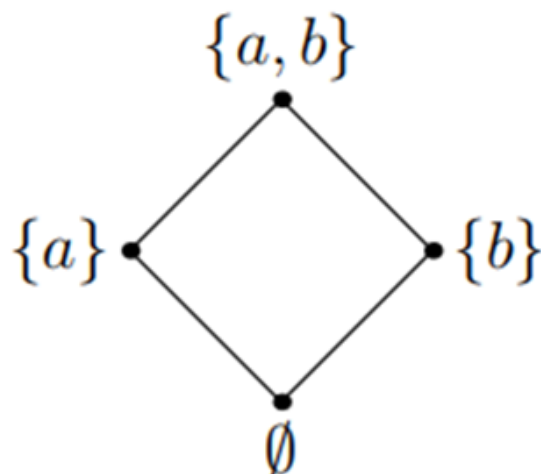
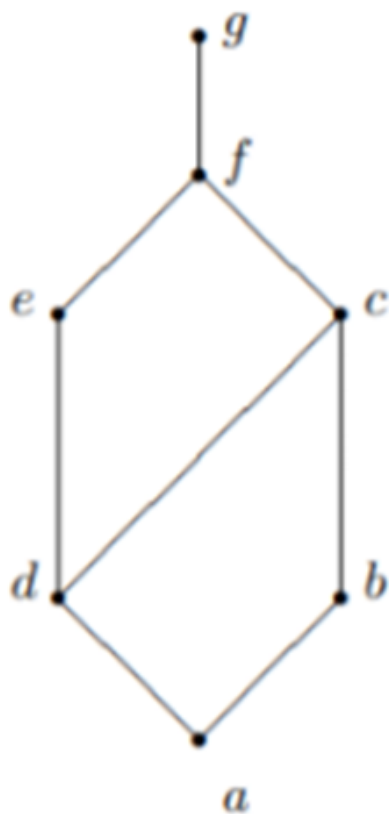


Figure: Hasse diagram of $\langle \text{pow}(\{a, b\}), \subseteq \rangle$

1. Consider the following Hasse diagram



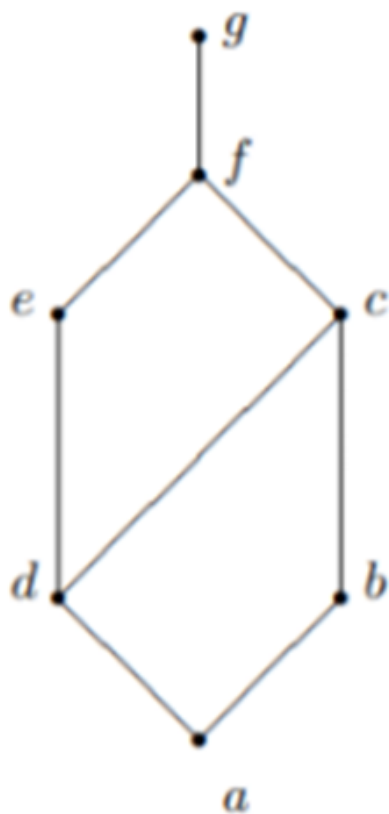
POSET

Check if it is
Lattice or not!

Every pair of
elements have
GLB & LUB.

1. Consider the following Hasse diagram

(—, —)
 ↓
 1 way
 ↓
 1 choice



How many pairs of elements!

⇒ 49 Pairs

a, a

$$a \vee a = a$$

$$a \wedge a = a$$

a, b

$$a \vee b = b$$

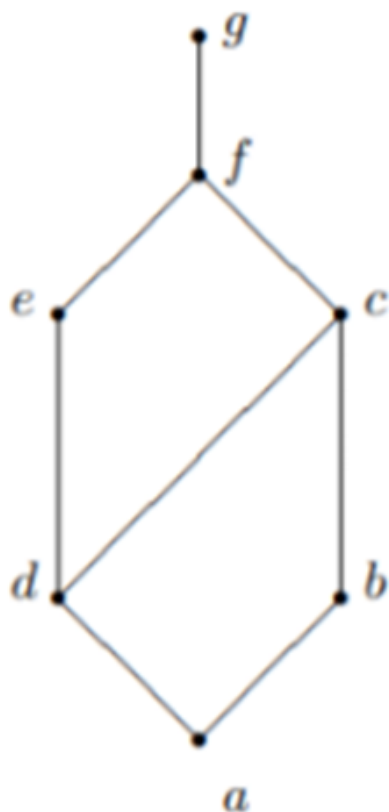
$$a \wedge b = a$$

b, a

$$b \vee a = b$$

$$b \wedge a = a$$

1. Consider the following Hasse diagram



$$a \vee a = a$$

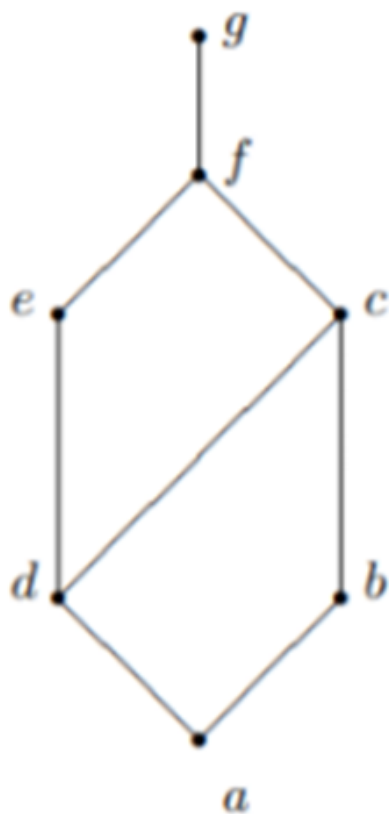
$$b \vee b = b$$

$$c \wedge c = c$$

$$f \wedge f = f$$

Nothing
to
check
Here.

1. Consider the following Hasse diagram



for Comparable
elements :

$$d \vee f = f$$

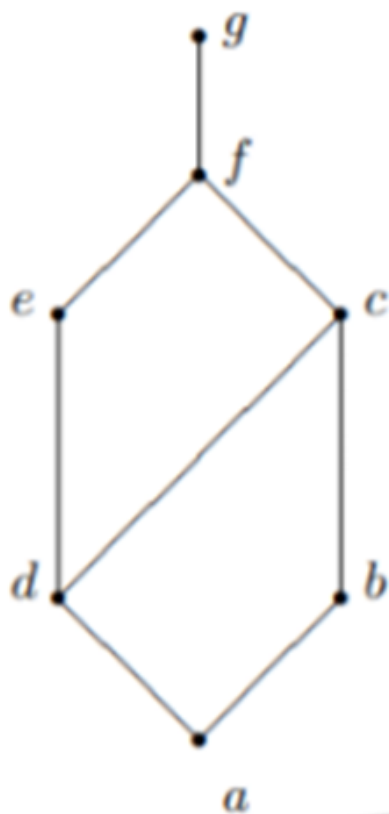
$$d \wedge f = d$$

$$e \vee g = g$$

$$e \wedge g = e$$

Nothing
to
check

1. Consider the following Hasse diagram



We only have to
check GLB, LUB
for Incomparable

pairs:

with b: \Rightarrow incomparable:

d, e

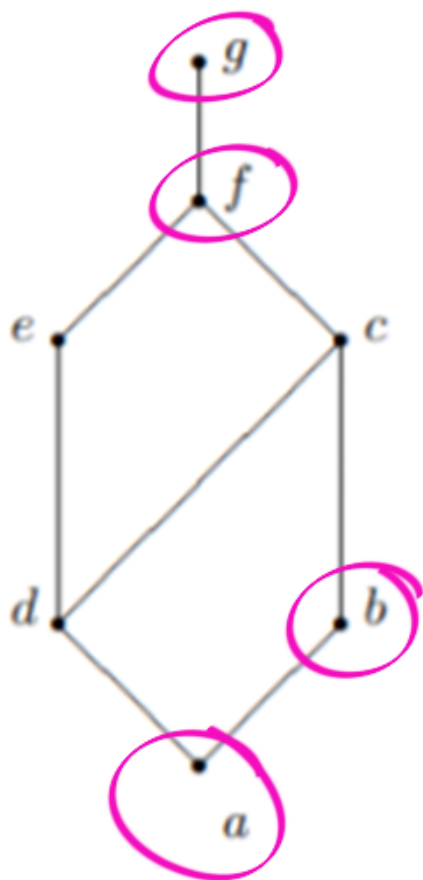
$$b \vee d = c$$

$$b \wedge d = a$$

$$b \vee e = e$$

$$b \wedge e = a$$

1. Consider the following Hasse diagram



We only have to
check GLB, LUB
for Incomparable

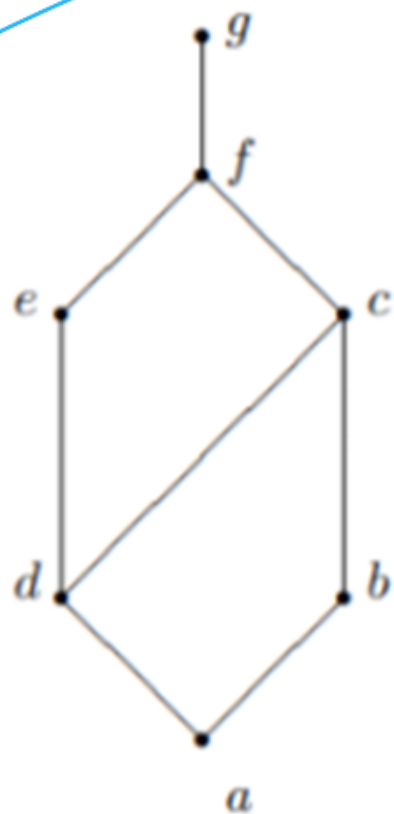
Pairs:

only e, c remaining:

$$e \vee c = f$$

$$e \wedge c = d$$

1. Consider the following Hasse diagram



a
Lattice
✓

We only have to
check GLB, LUB
for Incomparable
pairs:

a Lattice ✓

Q: To check if a given POSET (of n elements) is lattice or not, we have to check for EVERY Pair of elements to see if every pair of elements has LUB, GLB. It will take a lot of time as number of pairs of elements is n^2 . Can we do better than this??

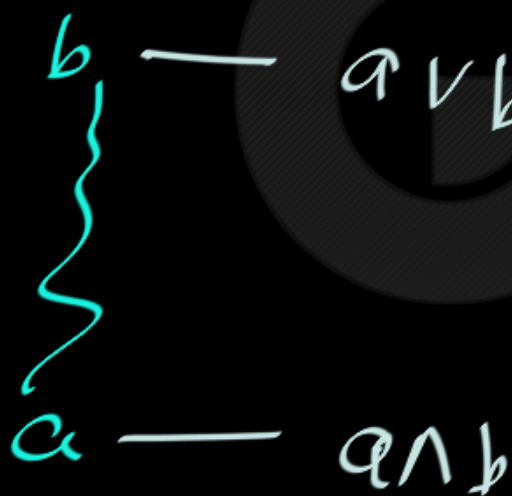
Q: To check if a given POSET (of n elements) is lattice or not, we have to check for EVERY Pair of elements to see if every pair of elements has LUB, GLB. It will take a lot of time as number of pairs of elements is n^2 . Can we do better than this??

Ans: Check only pairs of incomparable elements. ✓

Because Pair of Comparable elements ALWAYS have GLB, LUB in every POSET. So, there is No need of checking for them.

Note:

Two "Comparable" elements a, b
ALWAYS have GLB, LUB.



"Lattice"

Poset in which "EVERY two"
elements have LUB, GLB.

Lattice: a Poset in which
Every two Non-Comparable
Elements have GLB, LUB.
(i.e. GLB, LUB exists for Every
two Non-Comparable elements)

To check for lattice :

① Don't check $\{a, a\}$ type of

Pairs :

$$a \vee \bar{a} = a; \quad a \wedge \bar{a} = a$$

② Don't check Comparable elements



$$a \wedge b = a; \quad a \vee b = b$$

③ Just check Non-Comparable elements.

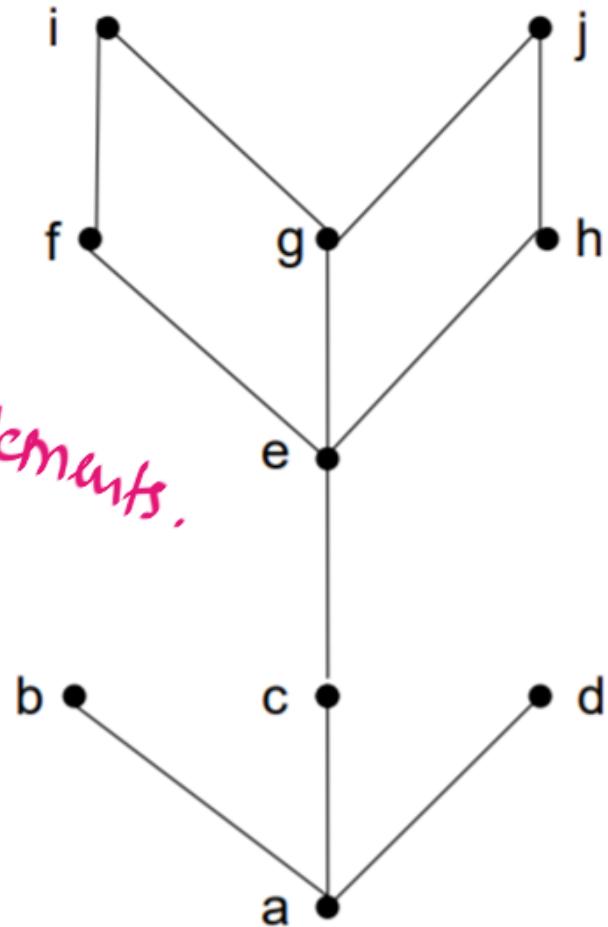
Lattices: Example 1

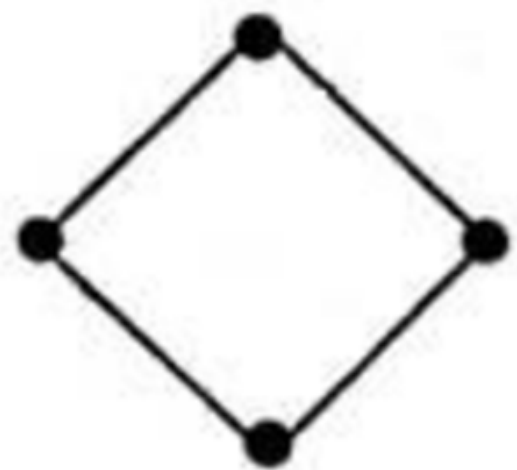
- Is the example from before a lattice?

LUB, GIB must exist for every pair of elements.

- No, because the pair $\{b,c\}$ does not have a least upper bound

$$LUB \{i,j\} = \underline{\underline{DNE}}$$

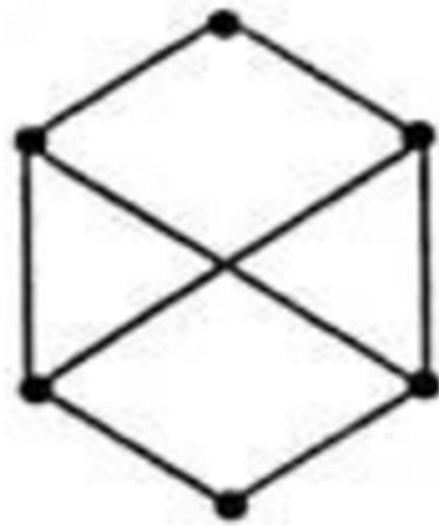




(i)



(ii)



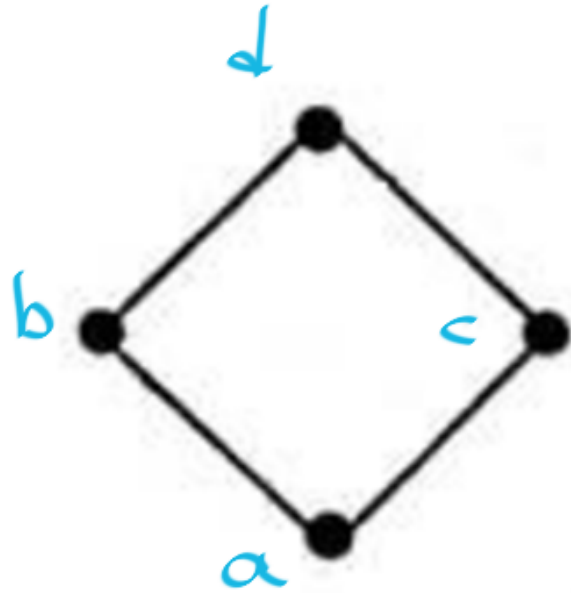
(iii)



(iv)

$$b \vee c = d$$

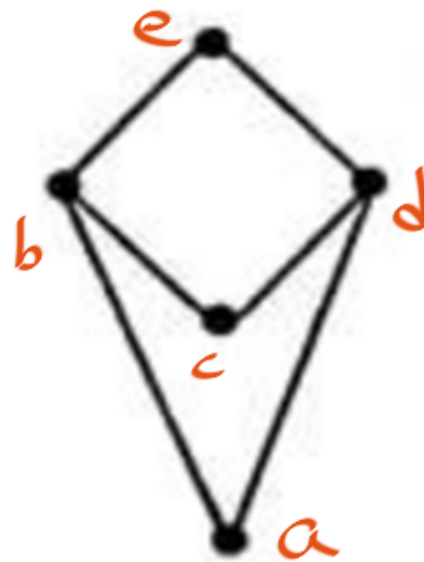
$$b \wedge c = a$$



(i)

Lattice ✓

$$c \wedge a = \text{DNE}$$

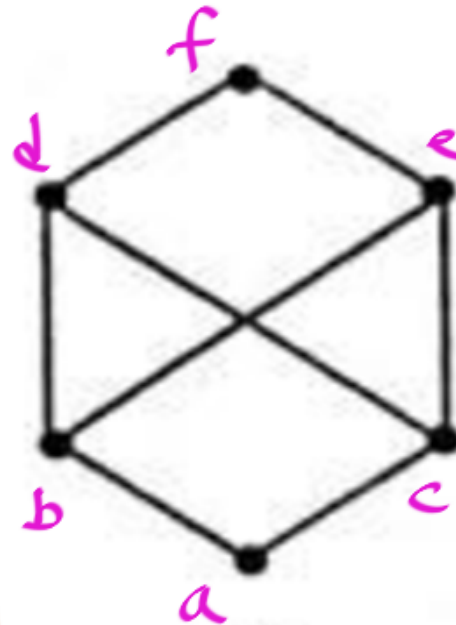


(ii)

Lattice ✗

$$b \vee c = \text{DNE}$$

$$d \wedge e = \text{DNE}$$



(iii)

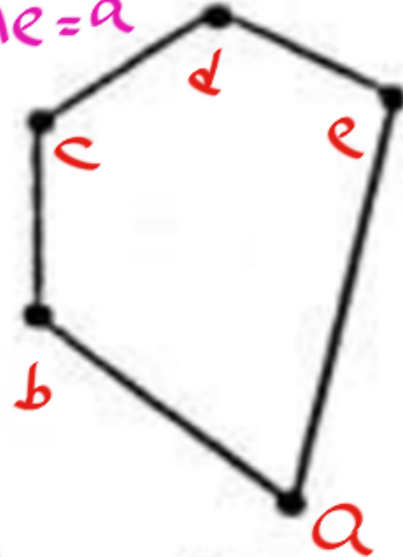
Lattice ✗

$$b \vee e = d$$

$$b \wedge e = a$$

$$c \vee e = d$$

$$c \wedge e = a$$



(iv)

Lattice ✓



Q: True / False?

Every Total Order Set is a Lattice?



Q: ~~True~~ / False?

Every Total Order Set is a Lattice?

True.

Every pair of elements is comparable.
No incomparable pair of elements

Every total order is a Lattice: Yes

↳ No pair of Non-Comparable elements.

~~$\forall a, b$~~ , a, b are Comparable

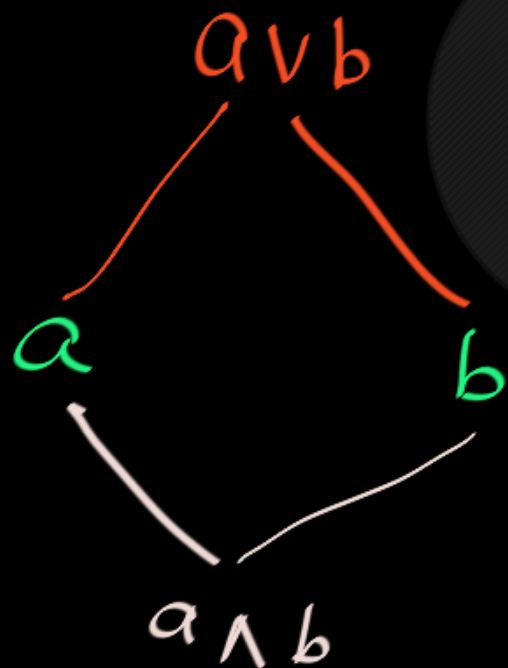
↳ $a \vee b = \text{exists}$; $a \wedge b = \text{exists}$



Properties satisfied
by **EVERY** Lattice

In EVERY Lattice :

$$\textcircled{1} \quad a R (a \vee b) \quad ; \quad (a \wedge b) R a$$



$$\left. \begin{array}{l} a R (a \vee b) \\ b R (a \vee b) \\ (a \wedge b) R a \end{array} \right\} \quad \left. \begin{array}{l} (a \wedge b) R b \\ (a \wedge b) R (a \vee b) \end{array} \right\}$$

In EVERY Lattice :

② Idempotent Property :

$$\left\{ \begin{array}{l} a \wedge a = a \\ a \vee a = a \end{array} \right.$$

$$a \wedge a \wedge a \wedge a = a$$

$$a \vee a \vee a \vee a = a$$

In EVERY Lattice :

③ Commutative Property:

$$\left\{ \begin{array}{l} x \vee y = y \vee x \\ x \wedge y = y \wedge x \end{array} \right.$$

$$\left\{ \begin{array}{l} a \vee b = b \vee a \\ a \wedge b = b \wedge a \end{array} \right.$$

In EVERY Lattice :

④ Associative Property:

$$(a \vee b) \vee c = a \vee (b \vee c)$$

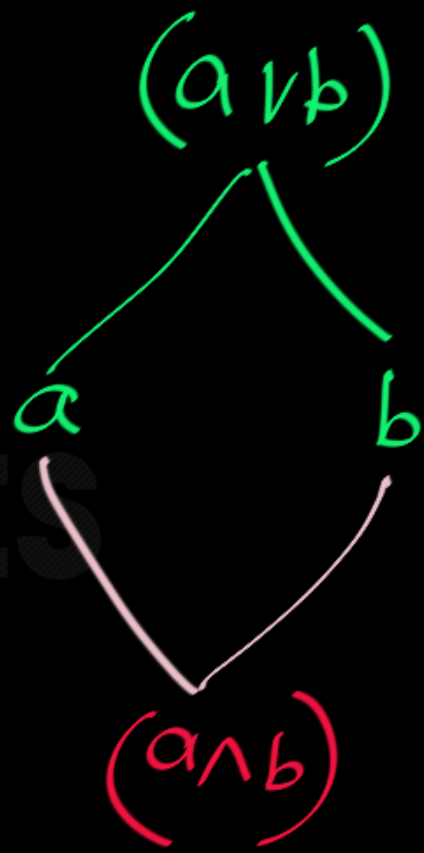
$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

In EVERY Lattice :

⑤ Absorption Law :

$$a \wedge (a \vee b) = a$$

$$a \vee (a \wedge b) = a$$

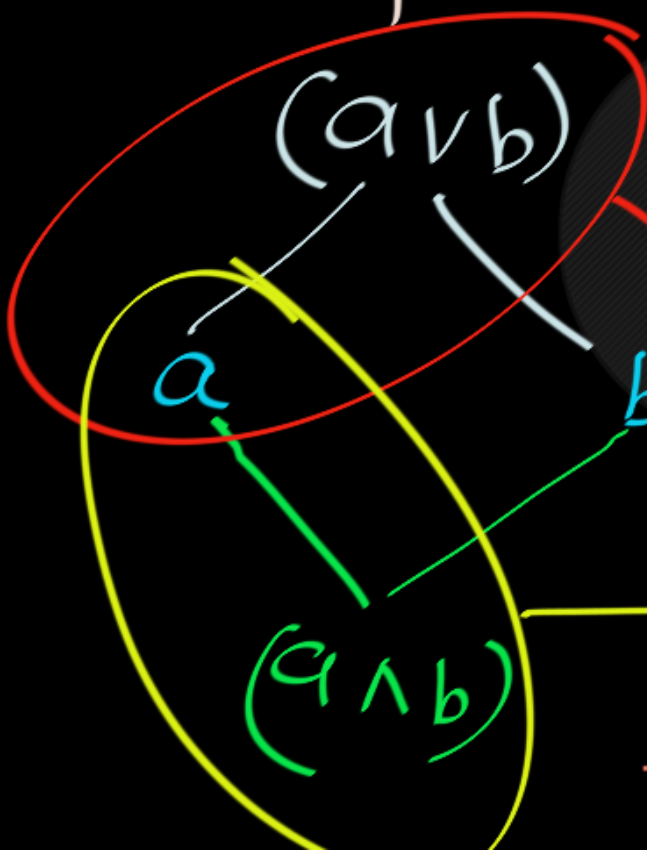


In EVERY Lattice :

Proof 1:

$$a \wedge (a \vee b) = a$$

$$a \vee (a \wedge b) = a$$



Comparable



Proposition

Any lattice has the following properties:

1. *Commutativity:* $a \cap b = b \cap a$ and $a \cup b = b \cup a$.
2. *Associativity:* $(a \cap b) \cap c = a \cap (b \cap c)$ and $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law:* $a \cap a = a$ and $a \cup a = a$.
4. *Absorption law:* $(a \cup b) \cap a = a$ and $(a \cap b) \cup a = a$.



Let L be a lattice. Define the **meet** (\wedge) and **join** (\vee) operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements $x, y,$ and z of a lattice L .

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (**commutative laws**)

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ (**associative laws**)

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (**absorption laws**)

d) $x \wedge x = x$ and $x \vee x = x$ (**idempotent laws**)



Special Types of Lattices

Types of Lattices:

1. Bounded Lattices
2. Complemented Lattice
3. Distributive Lattice
4. Boolean lattice

Properties Satisfied by EVERY Lattice:

PROPOSITION 7.2 - 2: Basic Operational Properties of Meet and Join

Let \mathcal{A} be a lattice with order relation \leq . Then the following properties hold:

- Commutativity:** $x \wedge y = y \wedge x$; $x \vee y = y \vee x$.
- Associativity:** $(x \wedge y) \wedge z = x \wedge (y \wedge z)$; $(x \vee y) \vee z = x \vee (y \vee z)$.
- Idempotence:** $x \wedge x = x$; $x \vee x = x$.
- Absorption:** $x \wedge (x \vee y) = x$; $x \vee (x \wedge y) = x$.

Are these Properties Satisfied by **EVERY** Lattice??

$$a \vee 0 = a$$

$$a \wedge 1 = a$$

identity

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

distributivity

$$a \vee \neg a = 1$$

$$a \wedge \neg a = 0$$

complements

NO. Some lattices satisfy these properties, Some don't.

Are these Properties Satisfied by **EVERY** Lattice??

$$a \vee 0 = a$$

$$a \wedge 1 = a$$

identity

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

distributivity

$$a \vee \neg a = 1$$

$$a \wedge \neg a = 0$$

complements

Bounded Lattice \iff Identity property

Distributive Lattice \iff Distributive property

Complemented Lattice \iff Complement property

Boolean Lattice \equiv Bounded & Distributive & Complemented

Types of Lattices:

1. Bounded Lattices \equiv Identity property
2. Complemented Lattice \equiv Complement prop.
3. Distributive Lattice \equiv Distributive prop
4. Boolean lattice \equiv All above 3 must satisfy



Bounded Lattice

(Identity Property Satisfied)

DEFINITION 7.2 - 4: Bounded Lattices

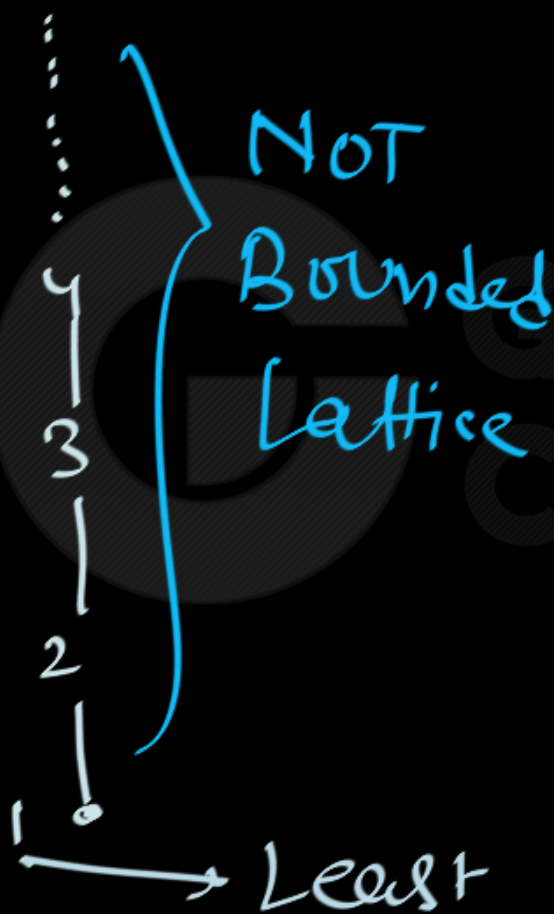
A lattice $\langle \mathcal{A}, \leq \rangle$ is **bounded** iff it has a minimum element and a maximum element. These are denoted by 0 and 1 respectively.

Least
(L/o)

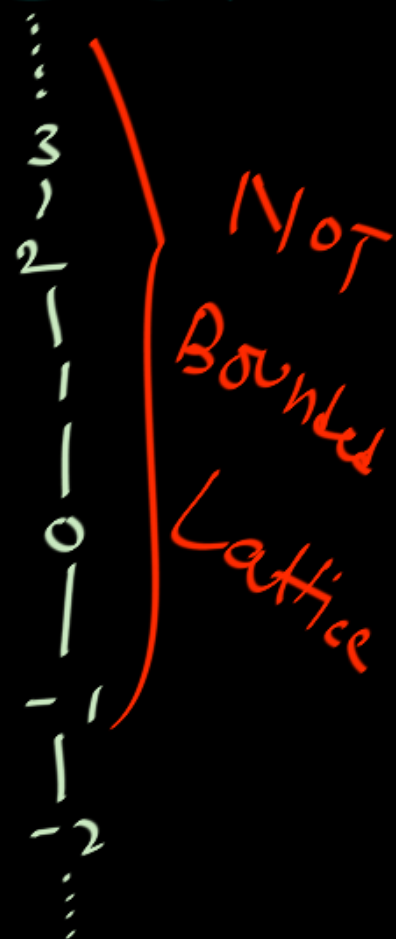
Greatest
(G/o)

Infinite Lattice which is NOT Bounded:

(\mathbb{N}, \leq)
No maximum element



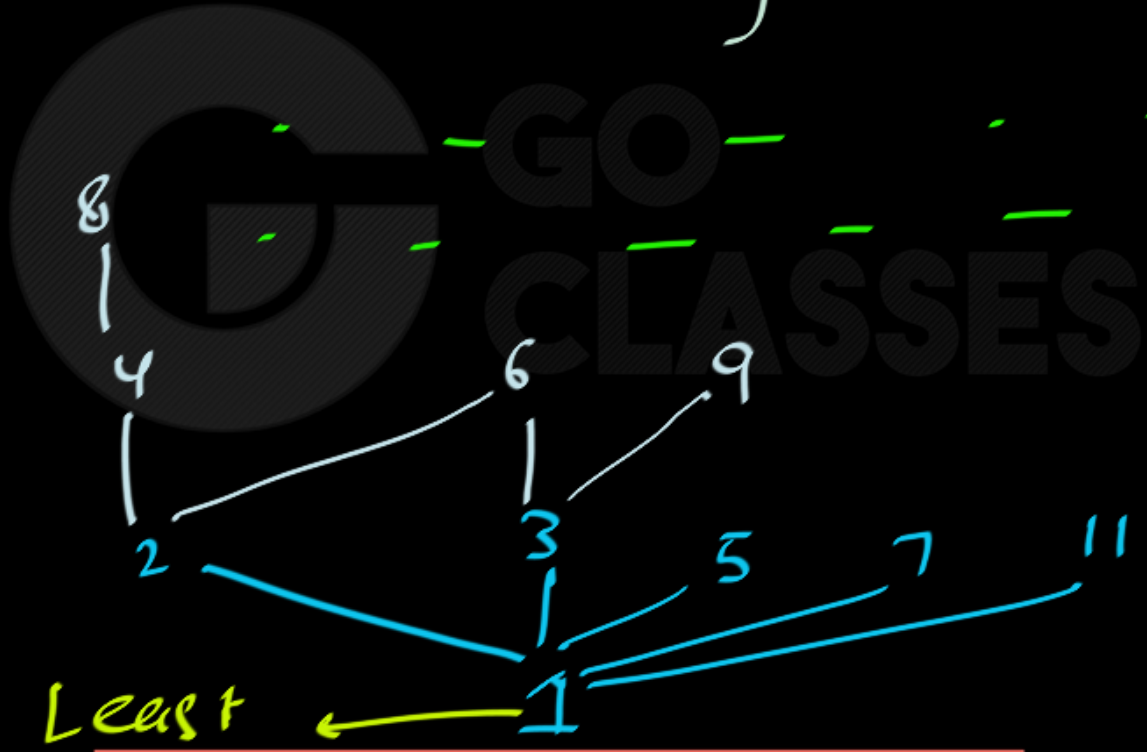
(\mathbb{Z}, \leq)
No Least
No Greatest



Infinite Lattice which is NOT Bounded:

$(N, |)$
 No
 Greatest
 element

$$N = \{1, 2, 3, \dots\}$$



This
 Lattice
 is
 NOT
 bounded

Infinite Lattice which is Bounded:

$$([0, 1], \leq)$$

Real line
interval
from 0 to 1
(included)

1 — Greatest

0 — Least

→ Tset → Lattice

Infinite Lattice which is Bounded:

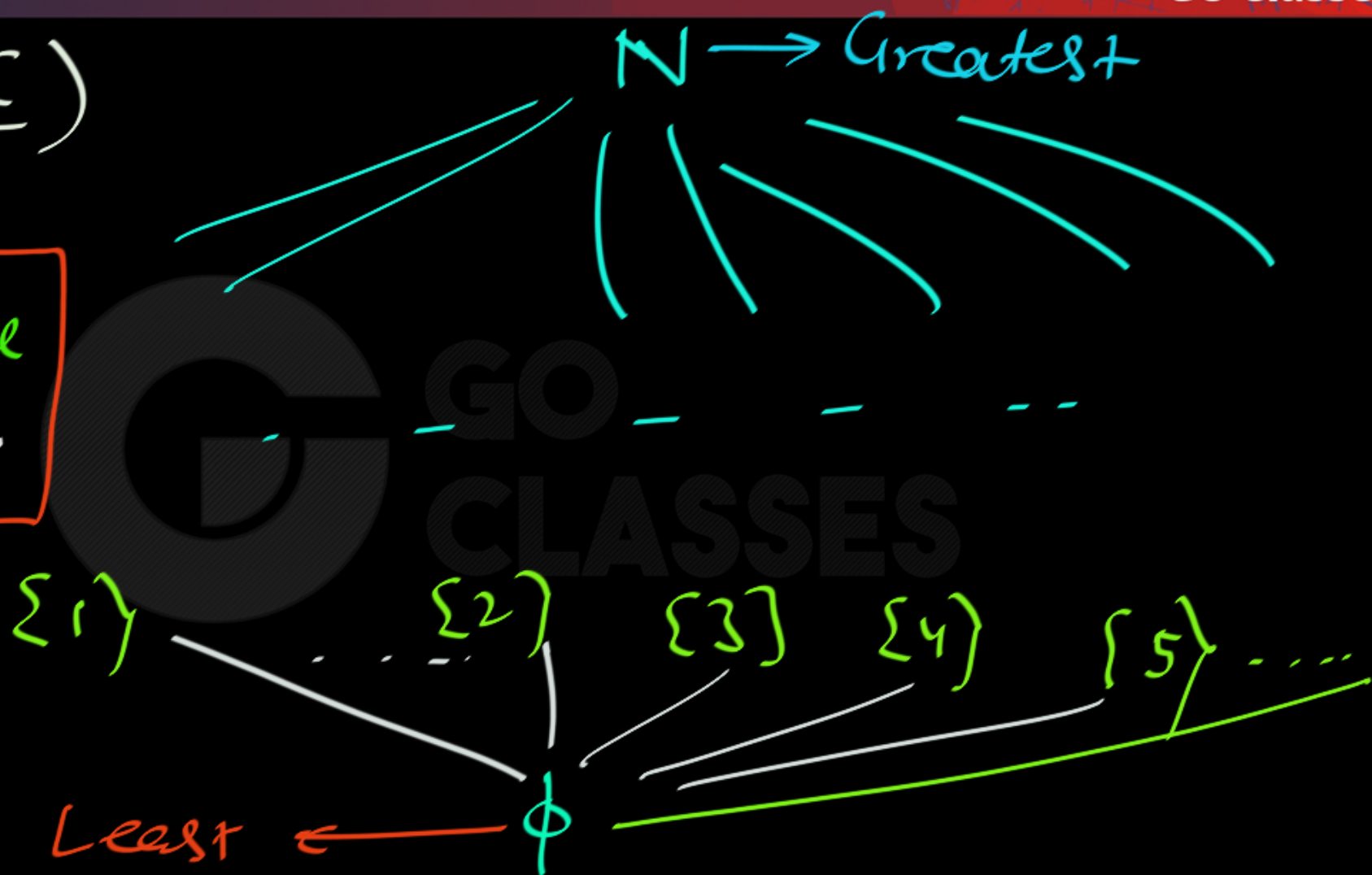
$$\left(\underline{P(N)}, \underline{\subseteq} \right)$$

subset

$$P(N) = \left\{ S \mid S \subseteq N \right\}$$

$(P(N), \subseteq)$

$\phi \subseteq \text{everyone}$
 $\text{everyone} \subseteq N$





NOTE:

Every Finite Lattice has a Greatest & a Least element...

So, Every Finite Lattice is Bounded.



NOTE:

Every Finite Lattice is Bounded.

An Infinite Lattice may or may not be Bounded.

Finite Lattice which is NOT Bounded:

NOT possible



Bounded Lattice : \rightarrow Greatest element $G/1$
 \rightarrow Least element $L/0$

$$\forall a \left\{ \begin{array}{l} a \vee G = G \\ a \wedge G = a \\ a \vee L = a \\ a \wedge L = L \end{array} \right\} \left\{ \begin{array}{l} a \vee 1 = 1 \\ a \wedge 1 = a \\ a \vee 0 = a \\ a \wedge 0 = 0 \end{array} \right\}$$

DEFINITION 7.2 - 4: Bounded Lattices

A lattice $\langle \mathcal{A}, \leq \rangle$ is **bounded** iff it has a minimum element and a maximum element. These are denoted by 0 and 1 respectively.

The extreme elements of bounded lattices interact with other elements of the lattice in the obvious ways, captured by the next proposition.

PROPOSITION 7.2 - 3: Extreme Elements in a Bounded Lattice

Suppose $\langle \mathcal{A}, \leq \rangle$ is a bounded lattice having minimum 0 and maximum 1, and let x be any element in \mathcal{A} . Then

$$\text{a) } 0 \vee x = x = x \vee 0; \quad 1 \wedge x = x = x \wedge 1$$

$$\text{b) } 0 \wedge x = 0 = x \wedge 0; \quad 1 \vee x = 1 = x \vee 1$$



Bounded Lattice & Identity Property

Identity property:

operation $\#$

Identity element "e"

$\forall a$

$$a \# e = a$$

$$e \# a = a$$

✓
boxed fixed for all

Identity property satisfied when $\exists e$.

Identity property:

School Life: operation: Addition (+)

Identity element $e = 0$

$$\forall x \left\{ \begin{array}{l} x + 0 = x \\ 0 + x = x \end{array} \right.$$

↓
fixes

Identity property:

School Life: operation: multiplication (\times)

Identity element $e = 1$

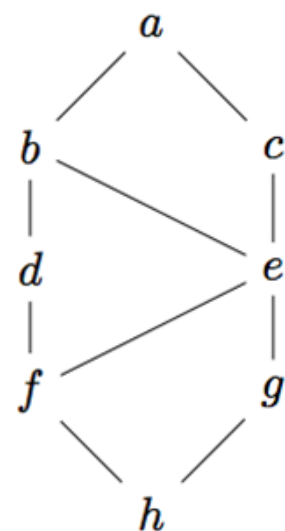
$$\forall x \left\{ \begin{array}{l} x \times 1 = x \\ 1 \times x = x \end{array} \right.$$

fixes

4. Let L be a lattice, defined by the Hasse diagram below.

operation: LUB

find Identity
Element for LUB.



$$a \vee x = a$$

$$b \vee x = b$$

$$h \vee x = h$$

$$g \vee x = g$$

$$x = h$$

fixed

Note:

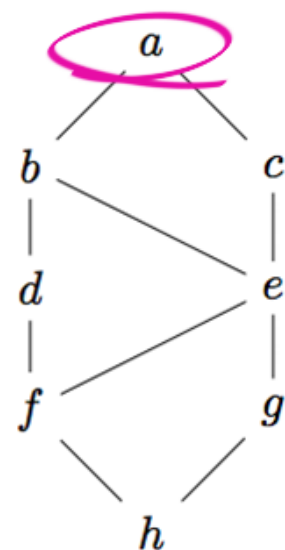
In any Lattice,

Identity element for LUB = Least Element

4. Let L be a lattice, defined by the Hasse diagram below.

operation: GLB

find Id element
for GLB



$$a \wedge y = a$$

$$b \wedge y = b$$

$$f \wedge y = f$$

$$h \wedge y = h$$

$$e \wedge y = e$$

$$y = a$$

fixed
for all

Note:

In any Lattice,

Identity element for GLB = Greatest Element

Note:

In any Lattice,

Identity element for GLB = Greatest Element

Identity element for LUB = Least element

Bounded Lattice:

a lattice with

G

$\&$

L

Identity for \wedge LB

Identity for
element for
LUB

Bounded Lattice:

- ① a lattice with $0 \in L$ ✓
- ② a lattice in which we have identity element for \wedge, \vee .
- ③ a lattice with Identity property.

Bounded Lattice

Identity
Property for
 \cap and \cup

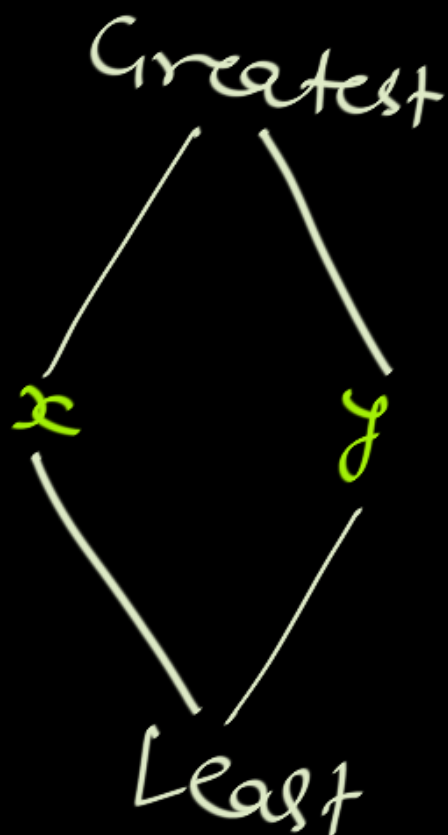


Complemented Lattice

Complement of an Element in a
Bounded Lattice :

"Complement of x " = y

iff $\left\{ \begin{array}{l} x \vee y = G = 1 \\ \text{and} \\ x \wedge y = L = 0 \end{array} \right\}$



Complement of an Element in a
Bounded Lattice :

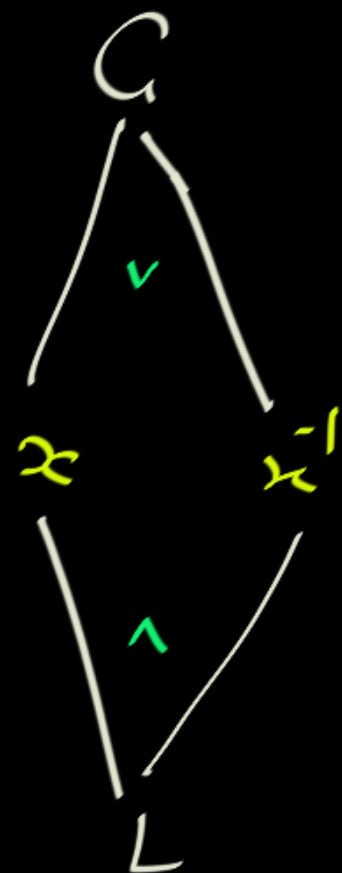
"Complement of x " = $x^{-1} = \overline{x} = y$

iff

$$x \vee y = \underline{\text{Greatest}}$$

and

$$x \wedge y = \underline{\text{Least}}$$



Complemented Lattice:

a Lattice in which Every element has at least one Complement.

DEFINITION 7.2 - 4: Bounded Lattices

A lattice $\langle \mathcal{A}, \leq \rangle$ is **bounded** iff it has a minimum element and a maximum element. These are denoted by 0 and 1 respectively.

DEFINITION 7.2 - 5: Complements in a Bounded Lattice

Suppose $\langle \mathcal{A}, \leq \rangle$ is a bounded lattice with minimum 0 and maximum 1. A **complement** of an element x is an element z such that $x \wedge z = 0$ and $x \vee z = 1$.

DEFINITION 7.2 - 6: Complemented Lattices

A bounded lattice $\langle \mathcal{A}, \leq \rangle$ is **complemented** iff every element has a complement.

Complemented lattices (cont'd)

$$\bar{a} = ?$$

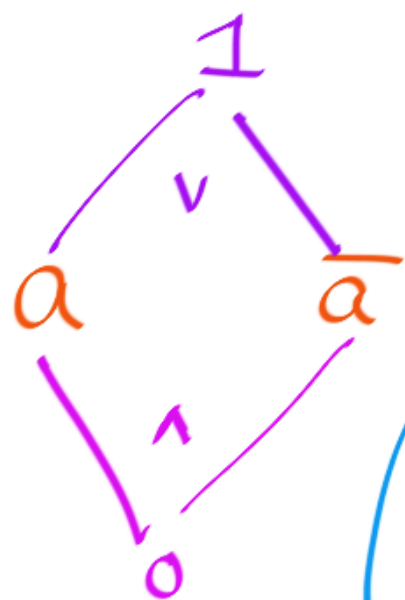
$$\bar{a} = b ?$$

$$a \vee b = 1 \checkmark$$

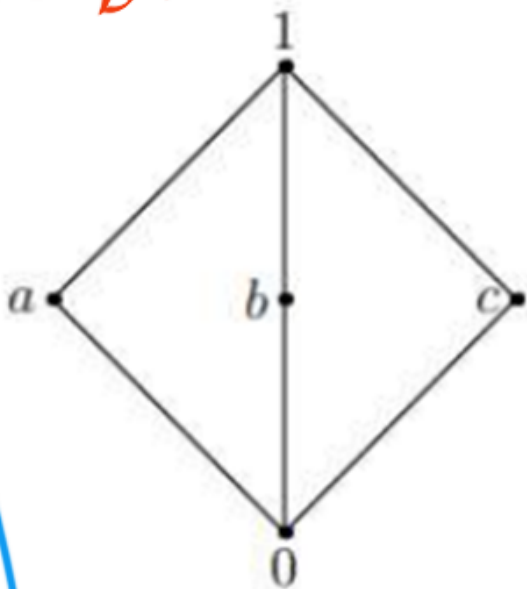
\neq

$$a \wedge b = 0 \checkmark$$

So, $\bar{a} = b$.



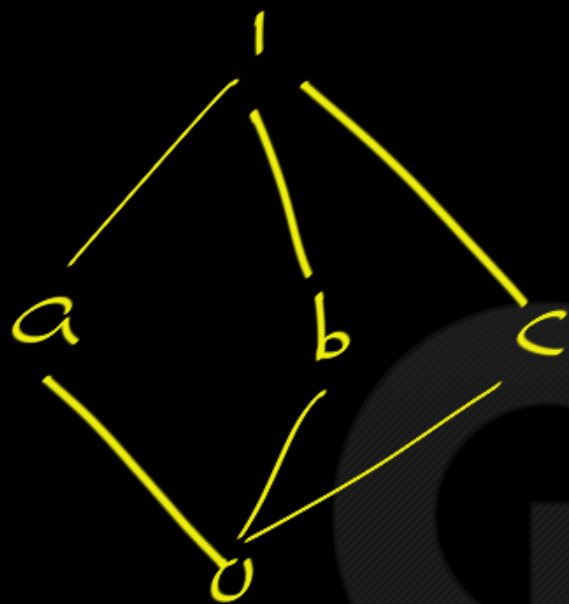
$$\begin{aligned} \bar{1} &= 0 \\ \bar{0} &= 1 \\ \bar{a} &= b, c \\ \bar{b} &= a, c \\ \bar{c} &= a, b \end{aligned}$$



$$\bar{a} = c \checkmark$$

$$\left\{ \begin{aligned} a \vee c &= 1 \\ a \wedge c &= 0 \end{aligned} \right\}$$

Figure: A complemented lattice where some complements are not unique



Is $\bar{a} = a$?

$$a \vee a = a \neq 1$$

$$a \wedge a = a \neq 0$$

So, $\bar{a} \neq a$.

Note:

$$\bar{a} = b \quad \text{iff}$$

$$a \vee b = U$$

$$a \wedge b = L$$

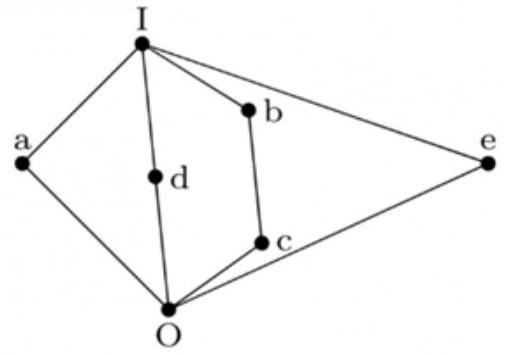
&

4.5.1 Lattice: GATE CSE 1988 | Question: 1vii [top](#)

<https://gateoverflow.in/91351>



The complement(s) of the element ' a ' in the lattice shown in below figure is (are) _____



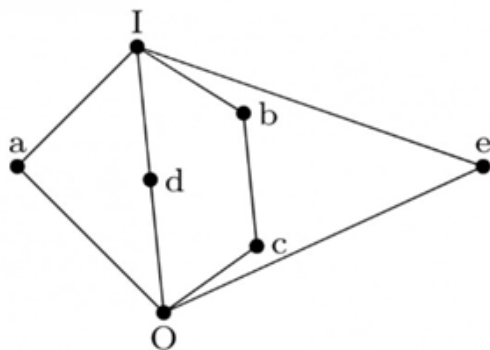
gate1988 descriptive lattice set-theory&algebra

4.5.1 Lattice: GATE CSE 1988 | Question: 1vii top<https://gateoverflow.in/91351>

The complement(s) of the element 'a' in the lattice shown in below figure is (are) _____

$$\bar{b} = ?$$

$$\bar{b} \neq c; \bar{c} \neq b$$



$$\bar{b} = c \text{ ! } \text{ No}$$

$$\left\{ \begin{array}{l} b \vee c = b \neq 1 \\ b \wedge c = c \neq 0 \end{array} \right.$$

gate1988 descriptive lattice set-theory&algebra

$$\bar{b} = e \text{ ? Yes}$$

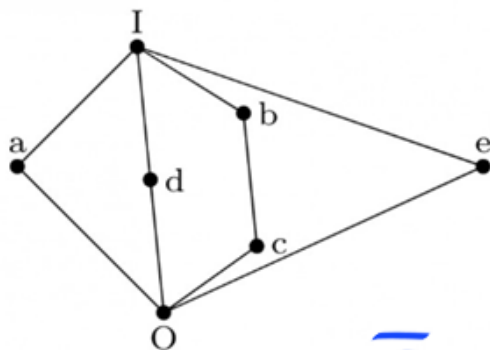
$$\left\{ \begin{array}{l} b \vee e = 1 \\ b \wedge e = 0 \end{array} \right.$$

4.5.1 Lattice: GATE CSE 1988 | Question: 1vii top

<https://gateoverflow.in/91351>



The complement(s) of the element 'a' in the lattice shown in below figure is (are) _____



$$\bar{a} = d$$

$$\left. \begin{aligned} a \vee d &= 1 \\ a \wedge d &= 0 \end{aligned} \right\} \checkmark$$

$$\bar{a} = b \checkmark$$

$$\left. \begin{aligned} a \vee b &= 1 \\ a \wedge b &= 0 \end{aligned} \right\} \checkmark$$

$$\bar{a} = e \checkmark$$

$$\left. \begin{aligned} a \vee e &= 1 \\ a \wedge e &= 0 \end{aligned} \right\} \checkmark$$

gate1988 descriptive lattice set-theory&algebra

$$\bar{a} = c \checkmark$$

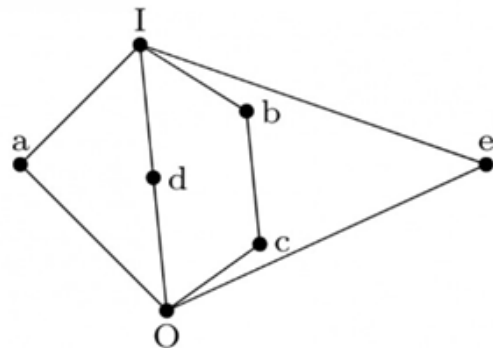
$$\left. \begin{aligned} a \vee c &= 1 \\ a \wedge c &= 0 \end{aligned} \right\} \checkmark$$

4.5.1 Lattice: GATE CSE 1988 | Question: 1vii [top](#)

<https://gateoverflow.in/91351>



The complement(s) of the element 'a' in the lattice shown in below figure is (are) _____



$\bar{a} = 0$? No

$a \vee 0 = a \neq I$

$\bar{a} = 1$? No.

$a \vee I = I$ ✓

But

$a \wedge I = a \neq 0$

gate1988 descriptive lattice set-theory&algebra

$\bar{a} = b, c, d, e$

$\bar{d} = a, b, c, e$

$\bar{e} = a, b, c, d$

$\bar{b} = ? = a, d, e$

$\bar{c} = ? = a, d, e$



Q:

Can a Total Order with More than 2 elements be a Complemented Lattice ??

Lattice

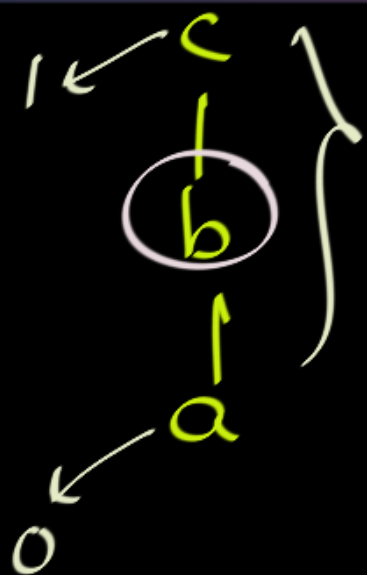
Q:

Can a Total Order with More than 2 elements be a Complemented Lattice??

No.

Every element has
at least one complement.





$$\bar{b} = ?$$

Does
not
Exist.

$$\bar{a} = c$$

$$\bar{c} = a$$

So NOT
a Comp. Lattice.

$$\bar{b} = b ? \text{ No}$$

$$b \vee b \neq 1$$

$$\bar{b} = a ? \text{ No}$$

$$b \vee a \neq 1$$

$$\bar{b} = c ? \text{ No}$$

$$b \wedge c \neq 0$$

$$\begin{array}{c} 1 \\ | \\ b \\ | \\ a \\ | \\ 1 \\ | \\ 0 \end{array}$$

$$\bar{a} = \text{DNE}$$

$$\bar{b} = \text{DNE}$$

$$\begin{array}{l} \bar{0} = 1 \\ \bar{1} = 0 \end{array}$$

NOT a
Comp. Lattice.



Distributive Lattice

A Lattice L is Distributive Lattice iff

$$\forall a, b, c \in L$$

$$\left\{ \begin{array}{l} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \end{array} \right.$$

DEFINITION 7.2 - 3: Distributive Lattice

A lattice \mathcal{A} is **distributive** iff the following **distributive laws** hold for any x, y , and z in \mathcal{A} :
 $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

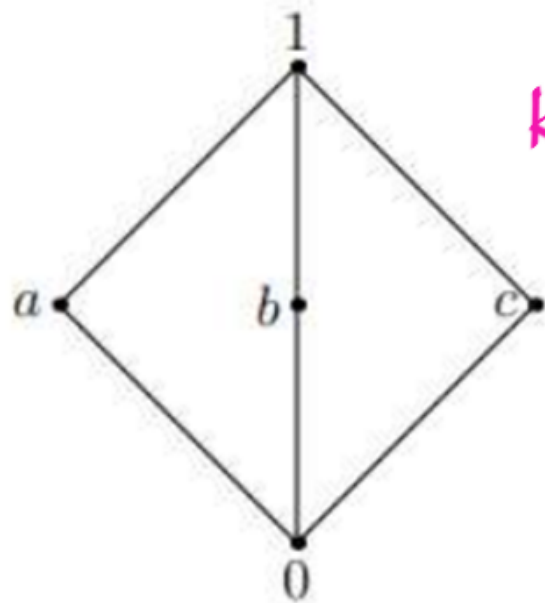
Definition (Distributive Lattice)

A lattice L is said to be distributive if the following conditions hold:

- ▶ $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$ for all $r, s, t \in L$;
- ▶ $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$ for all $r, s, t \in L$.

A non-distributive lattice

$$a \vee (b \wedge c) \not\equiv (a \vee b) \wedge (a \vee c)$$



kite lattice

Figure: $a \cap (b \cup c) \neq (a \cap b) \cup (a \cap c)$

A non-distributive lattice

$$a \vee (b \wedge c) \stackrel{?}{=} (a \vee b) \wedge (a \vee c)$$

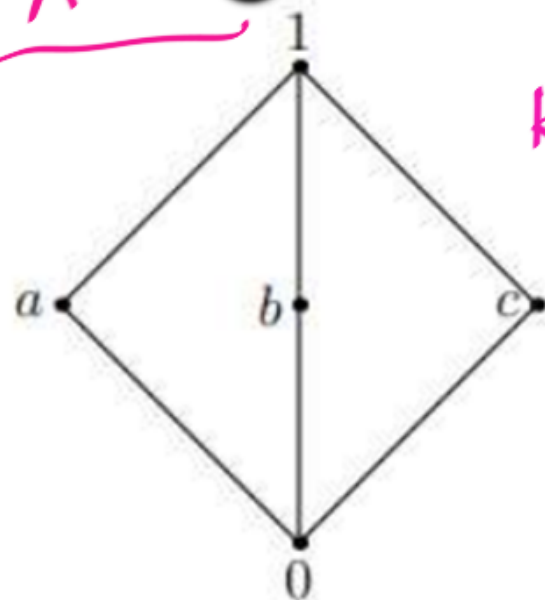
$$(a \vee 0)$$

a

\neq

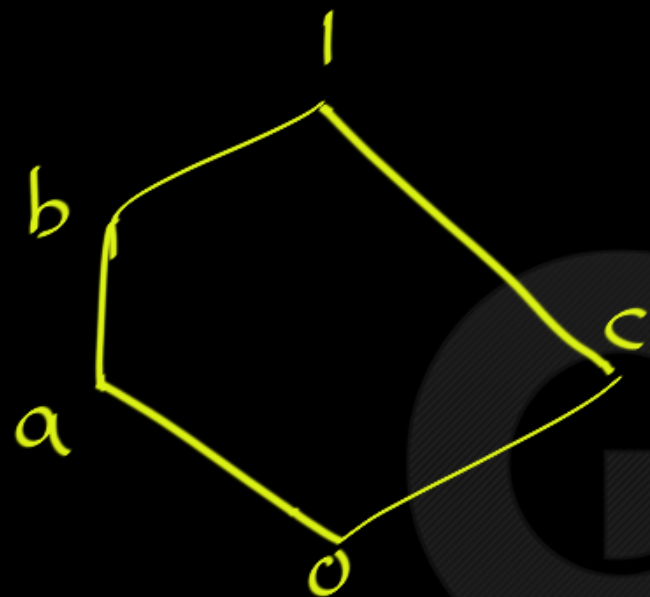
1

$$(1) \wedge (1)$$



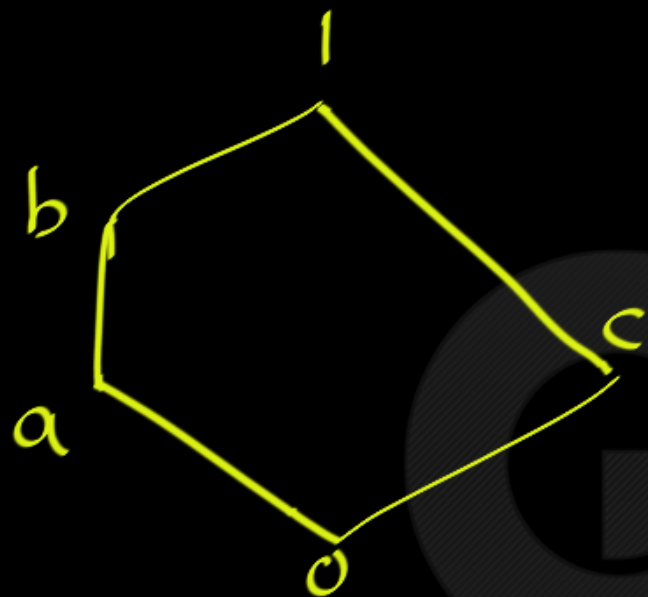
kite lattice

Figure: $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$



Pentagon Lattice

$$b \vee (a \wedge c) \stackrel{?}{=} (b \vee a) \wedge (b \vee c)$$



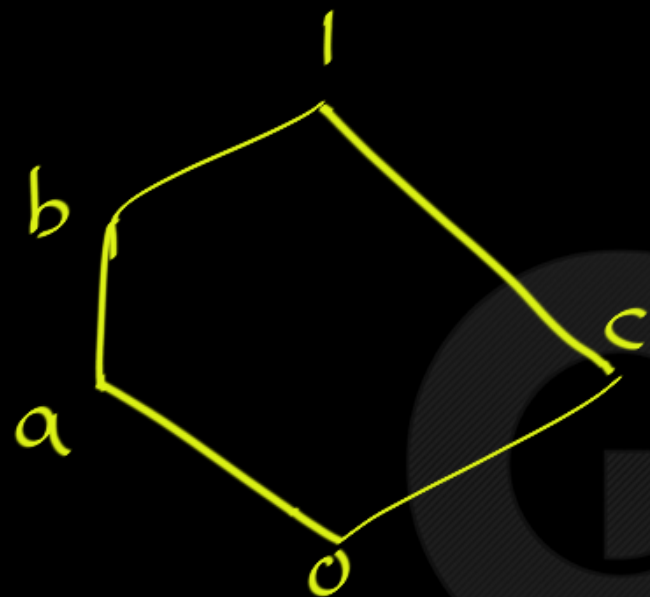
Pentagon Lattice

$$b \vee (a \wedge c) \stackrel{?}{=} (b \vee a) \wedge (b \vee c)$$

The diagram illustrates the distributive law with annotations:

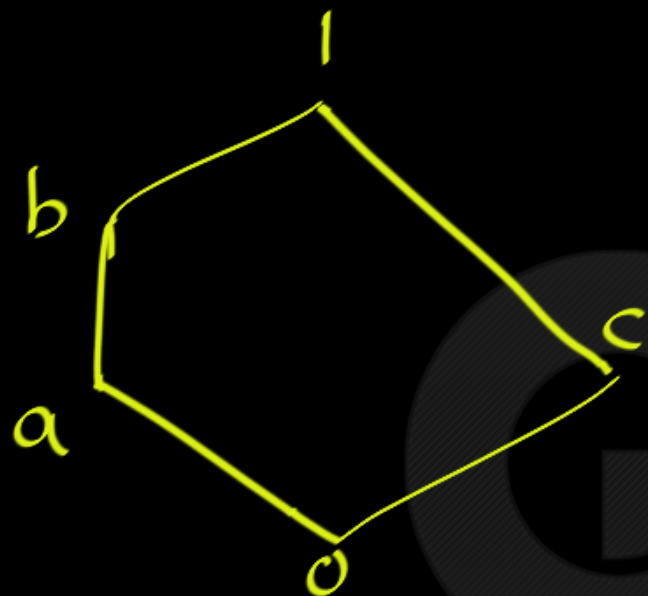
- A blue bracket under $a \wedge c$ has an arrow pointing to $(b \vee 0)$.
- A pink bracket under $(b \vee 0)$ has an arrow pointing to b .
- A blue bracket under $(b \vee a) \wedge (b \vee c)$ has an arrow pointing to b .
- Two orange brackets under $(b \vee a)$ and $(b \vee c)$ have arrows pointing to $b \wedge 1$.
- A blue bracket under $b \wedge 1$ has an arrow pointing to b .

The final result is $b = b$, with a double blue line indicating equality.



Pentagon Lattice

$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$



Pentagon Lattice

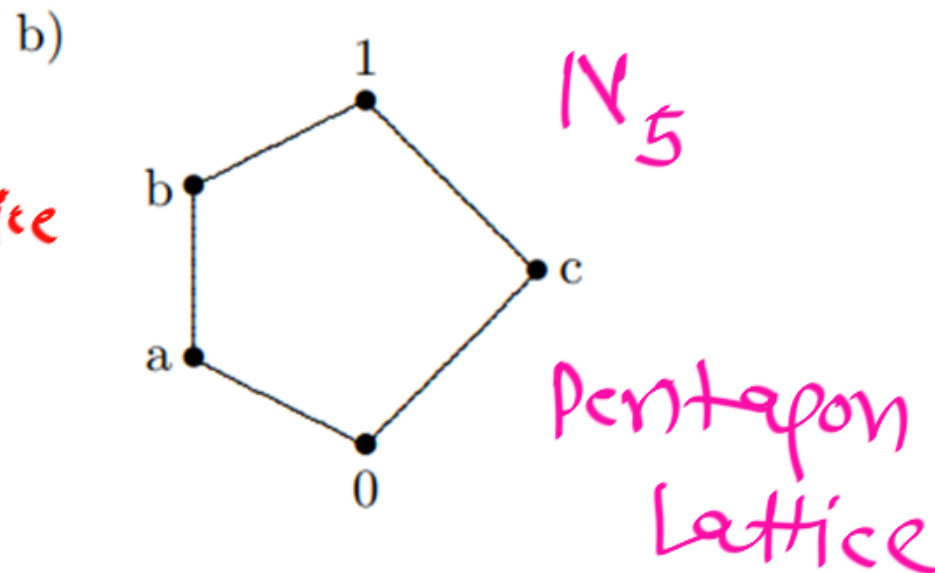
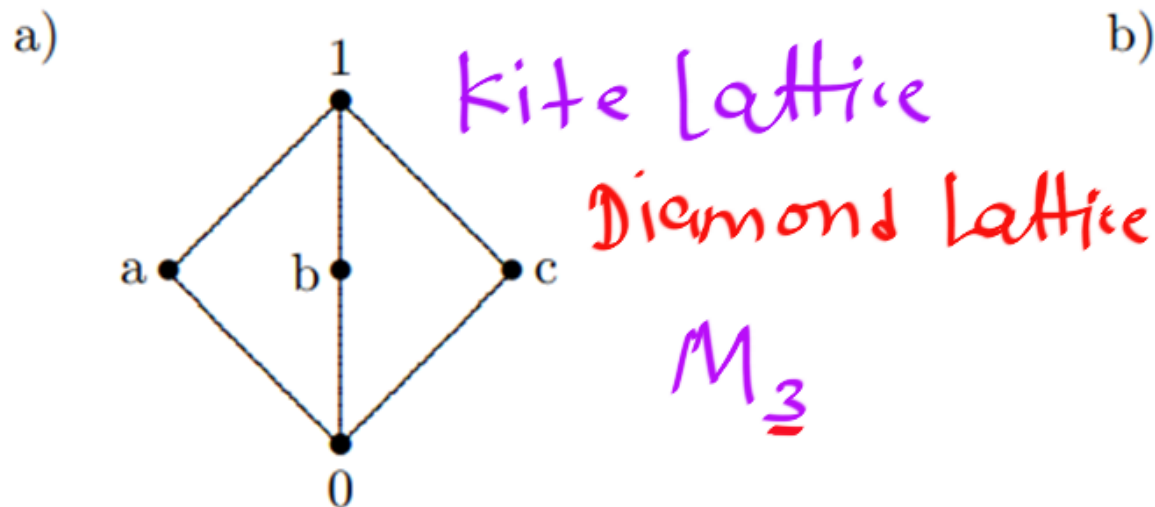
$$b \wedge (a \vee c) \stackrel{?}{=} (b \wedge a) \vee (b \wedge c)$$

$b \wedge (a \vee c) \neq (b \wedge a) \vee (b \wedge c)$

The diagram shows the distributive law being tested. On the left, $b \wedge (a \vee c)$ is evaluated. A red wavy line under $a \vee c$ points to $a \vee c$, which then points to b . This results in $b \wedge 1 = b$. On the right, $(b \wedge a) \vee (b \wedge c)$ is evaluated. Blue arrows point from $b \wedge a$ to a and from $b \wedge c$ to o . These are then combined as $a \vee o = a$. The final result is $b \neq a$.

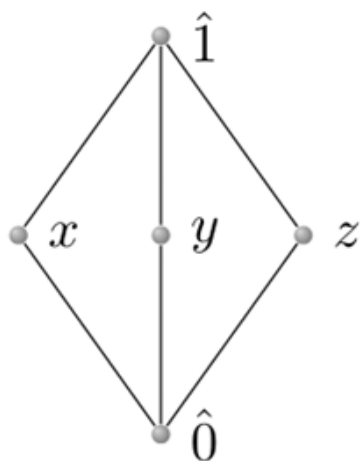
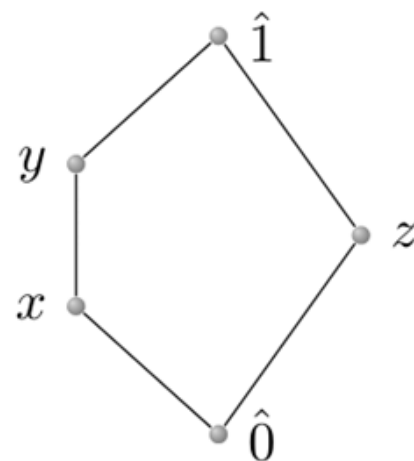
→ NOT Distributive Lattice.

Show that the following simple but significant lattices are not distributive.



Solution

- a) To see that the diamond lattice is not distributive, use the middle elements of the lattice:
 $a \wedge (b \vee c) = a \wedge 1 = a$, but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$, and $a \neq 0$.
 Similarly, the other distributive law fails for these three elements (see Exercise 22a).
- b) The pentagon lattice is also not distributive. See Exercise 22b.

Diamond lattice, M_3 Pentagon lattice, N_5

These examples characterize non-distributive lattices.

M_3, N_5 : Smallest non-Distributive lattices



Q:

Prove that

Every totally ordered set is a distributive lattice.



Q:

Prove that

Every **totally ordered set** is a distributive lattice.

→ Lattice

→ Distributive

Proof:

Total order: Take any a, b, c ($a \leq b \leq c$)



$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

The diagram shows the distributive law of disjunction over conjunction. The left side is $a \vee (b \wedge c)$. The right side is $(a \vee b) \wedge (a \vee c)$. Red arrows and brackets show the mapping: $a \vee b$ from the right side maps to $a \vee b$ in the left side, and $b \wedge c$ from the right side maps to $b \wedge c$ in the left side. Below the left side, a bracket under $(a \vee b)$ is labeled b , and a bracket under $(b \wedge c)$ is labeled b , with a double red line indicating their equivalence.

Proof:

Total order: Take any a, b, c ($a \leq b \leq c$)



$$b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c)$$

$(b \wedge c) = b$ (since $b \leq c$)
 $(b \wedge a) = a$ (since $a \leq b$)
 $\therefore b \wedge (a \vee c) = a \vee b = b$



Very

Important Theorem about Distributive Lattice

One Way theorem:

Distributive Lattice \longrightarrow Every element has at most 1 Complement

Some element has > 1 Complement

NOT Distributive



Theorem

For a distributive lattice L , the following condition hold.

The complement of each element, if exists, must be unique.

PROPOSITION 7.2 - 4: Uniqueness of Complements in Distributive Lattices

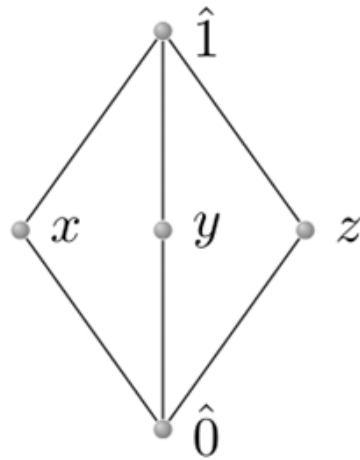
If $\langle \mathcal{A}, \leq \rangle$ is a bounded distributive lattice with minimum 0 and maximum 1, then complements are unique, provided they exist.



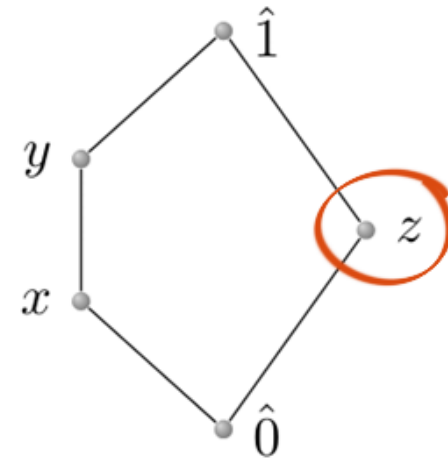
$\bar{x} = y, z$

↓

NOT Distributive



Diamond lattice, M_3



Pentagon lattice, N_5

$\bar{x} = z$

$\bar{y} = 0$

$\bar{z} = x, y$

These examples characterize non-distributive lattices.

NOT Distributive



Checking Distributive Lattice

Checking Distributive Lattice:

1. Every Lattice with ≤ 4 elements is Distributive.
2. Every Total Order Set is Distributive.

Checking Distributive Lattice:

3. If some element has > 1 complement, then Lattice is Not Distributive.
4. If Lattice contains M3 or N5 then it is Not Distributive.

Checking Distributive Lattice:

5.

FINITE Lattice in which Every element has EXACTLY One Complement, is Distributive.



Checking Distributive Lattice:

6.

Lattice with 5 elements is Distributive if it is NOT M3 or N5.

Guideline 5:

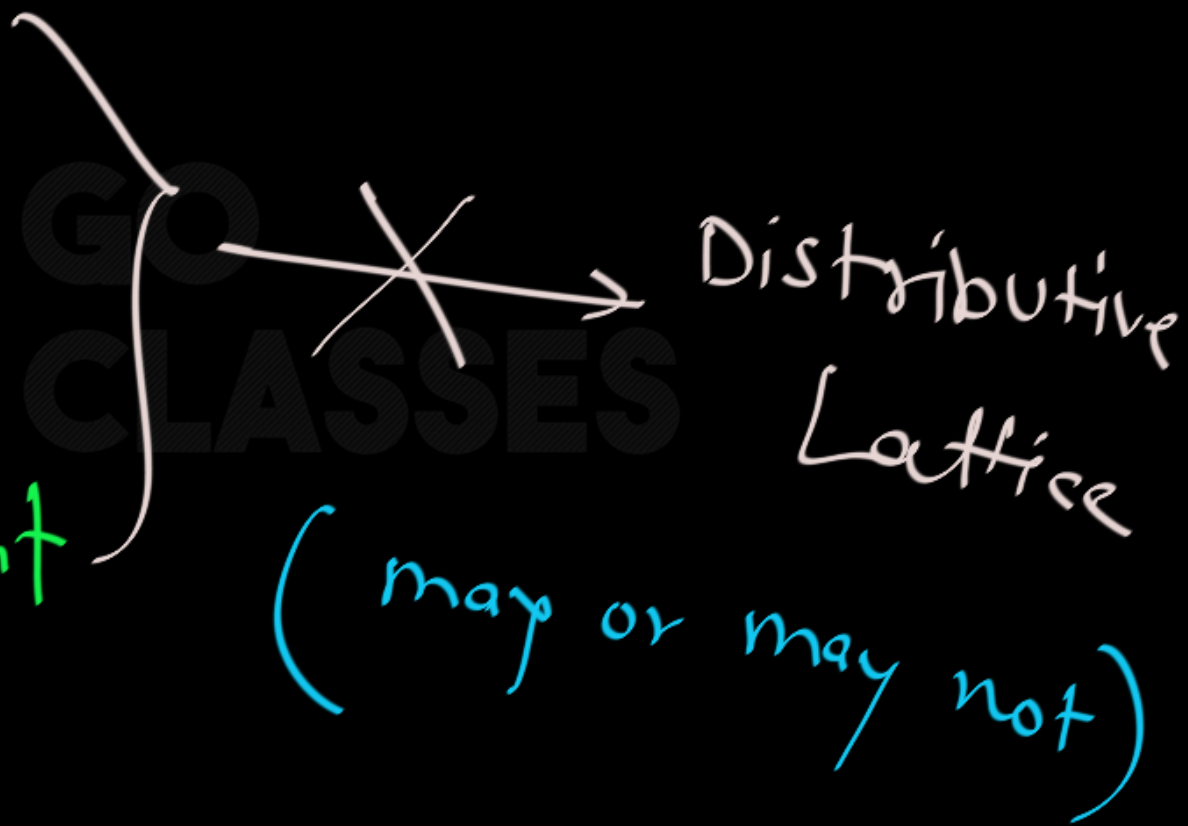
FINITE Lattice
in which EVERY
element has
Exactly 1 Complement

✓
→
✓
Distributive

Note:

Lattice

in which EVERY
element has
Exactly 1 complement



Note:

FINITE Lattice
in which EVERY
element has
 ≤ 1 Complement

~~→~~ Distributive Lattice

(may or may not)

Note:

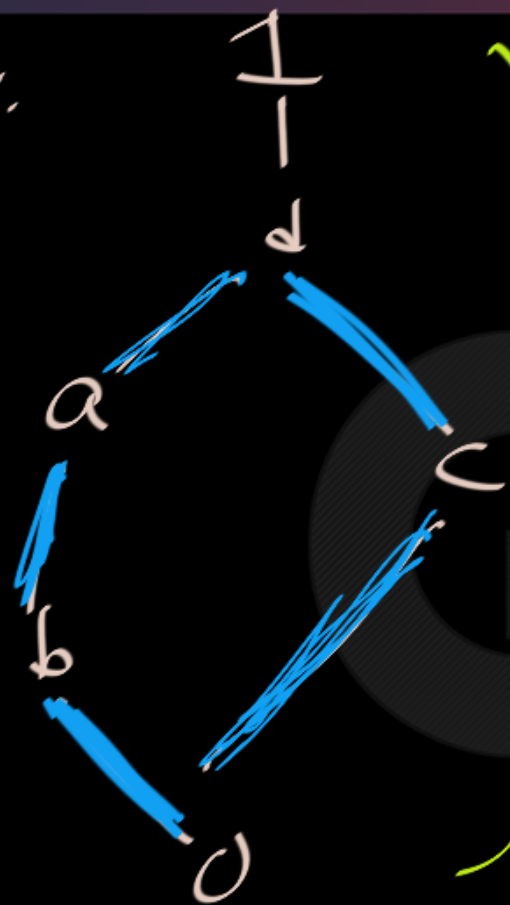
Distributive
Lattice



Every element has
 ≤ 1 Complement



Ex:



→ Lattice ✓

$$\bar{1} = 0$$

$$\bar{c} = \text{DNE}$$

$$\bar{0} = 1$$

$$\bar{a} = \text{DNE}$$

$$\bar{b} = \text{DNE}$$

Every element has ≤ 1 Complement

BUT Still NOT Dist.

NOT Distributive

$\bar{a} = c$?? No ($a \vee c = 1 \neq 1$)

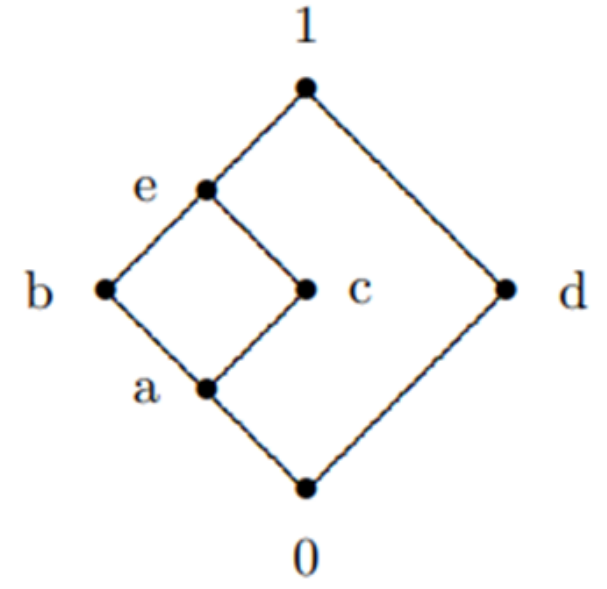
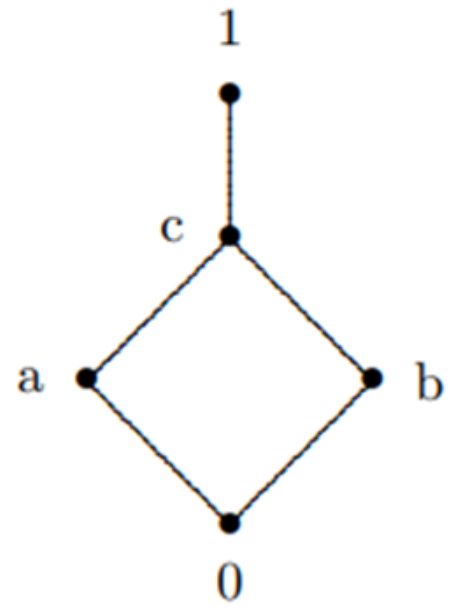
The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

EC

a.

*b.

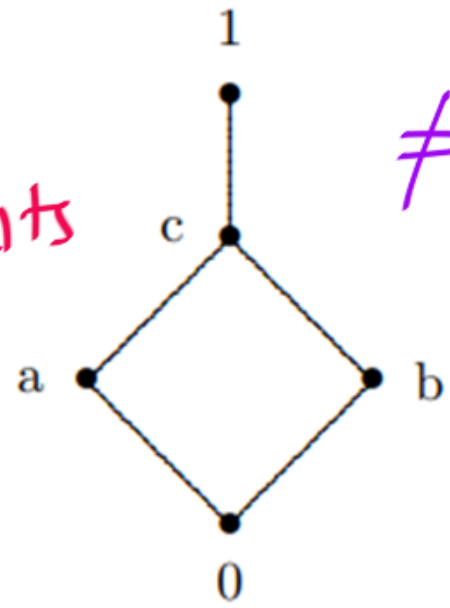


The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

EC

a.

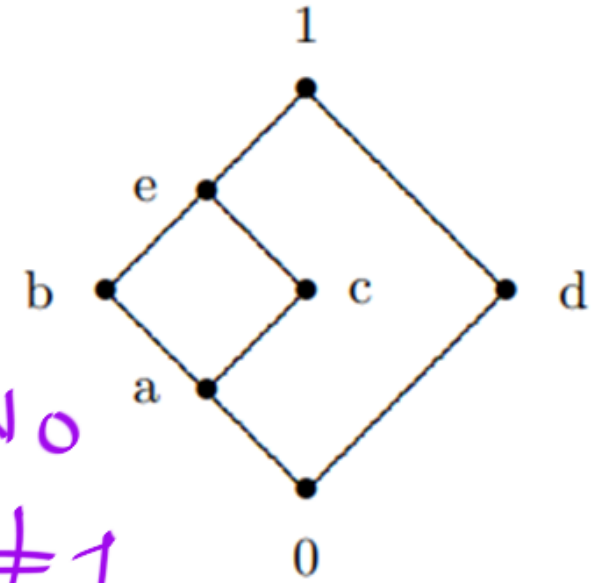


$\neq M_3$

$\neq N_5$

5 elements

*b.



$\bar{b} = d$

$\bar{b} = c$ No

$b \vee c = e \neq 1$

Distributive ✓

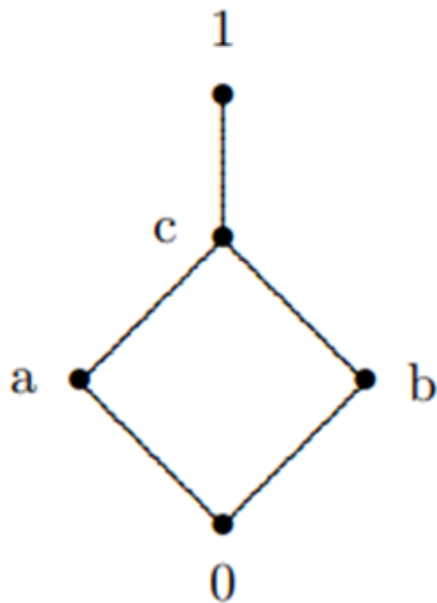
The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

EC

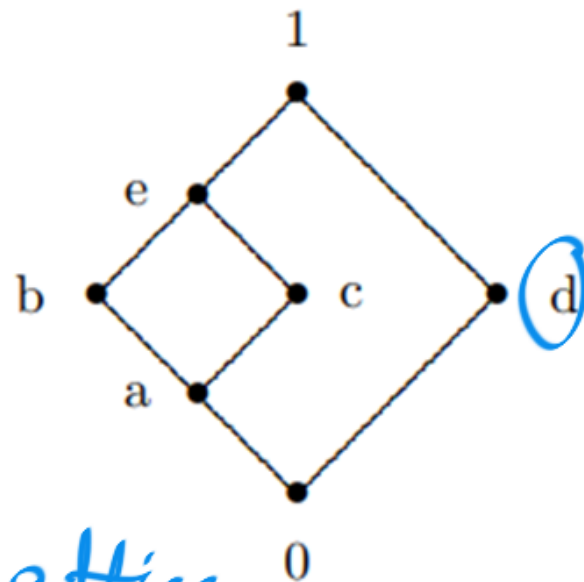
a.

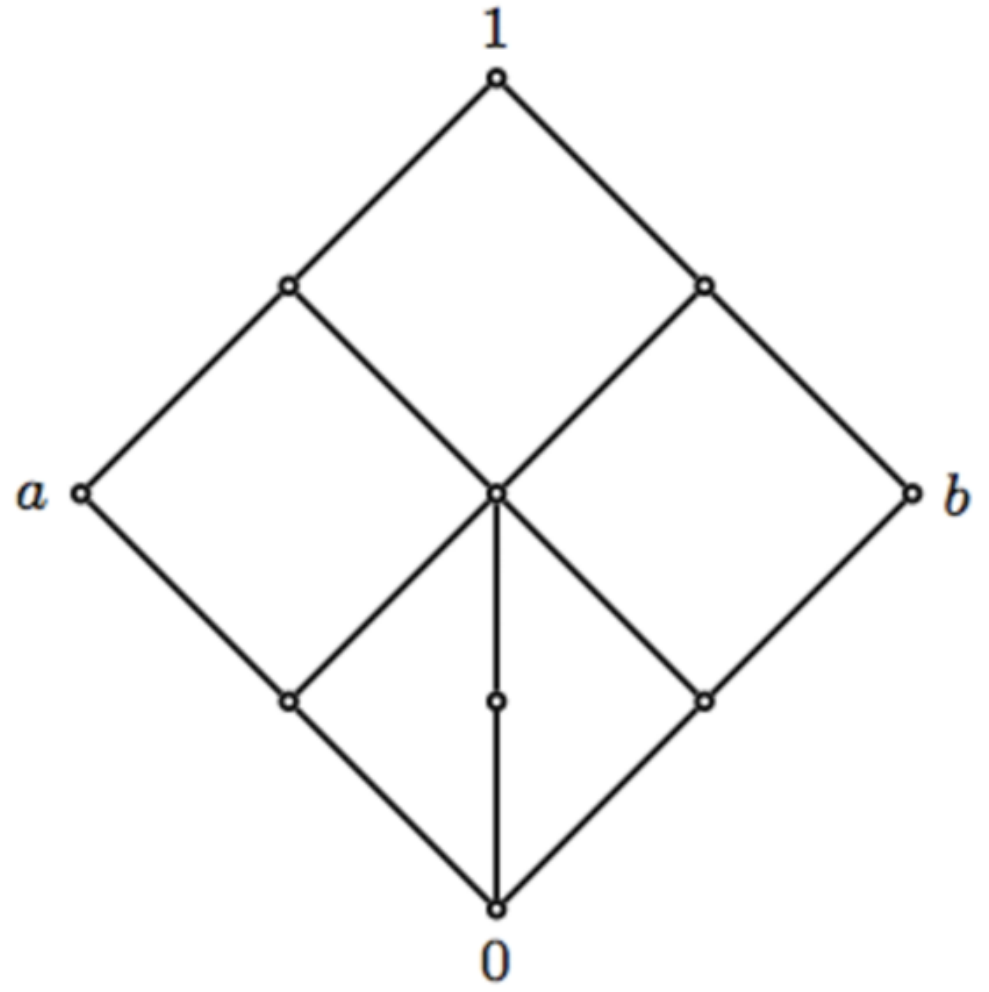
*b.

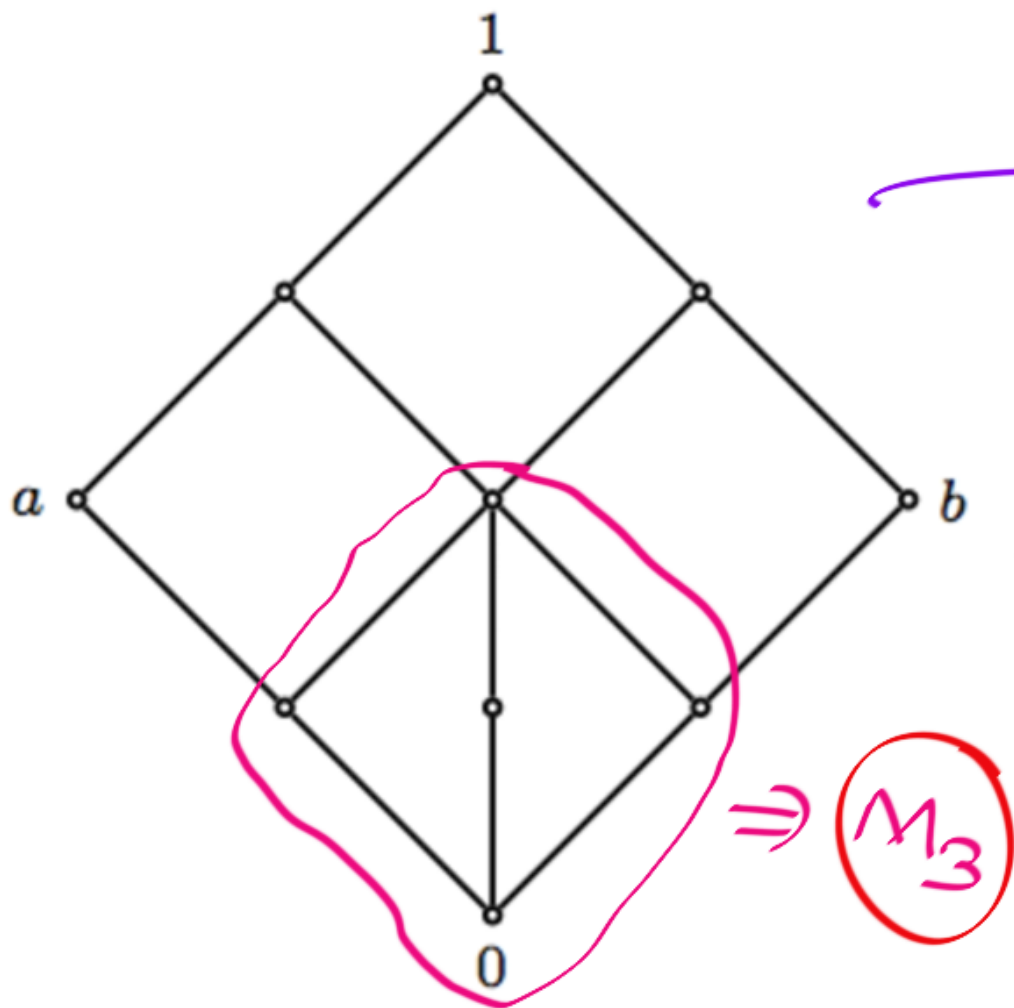


$\bar{b} = d$
 $\bar{d} = b, c$

NOT
 a Distributive Lattice







→ NOT a
Distributive
Lattice,

→ Every element
has ≤ 1 Complement

COROLLARY 1: Complemented Distributive Lattices Have Unique Complements

Every element in a complemented distributive lattice has a unique complement. The complement of x will be denoted by \bar{x} .

If Lattice L is Complemented & Distributive

Every element
has ≥ 1 Complement

Every element
has ≤ 1 Comp.



Boolean Lattice or Boolean Algebra

Boolean Lattice / Boolean Algebra :

Bounded & Complemented & Distributive

Boolean Lattice :

Complemented Distributive Lattice,

Definition

A **Boolean algebra** is a lattice with 0 and 1 that is distributive and complemented.

Example

Let A be a set. Then $\langle \text{pow}(A), \subseteq \rangle$ is a Boolean algebra.

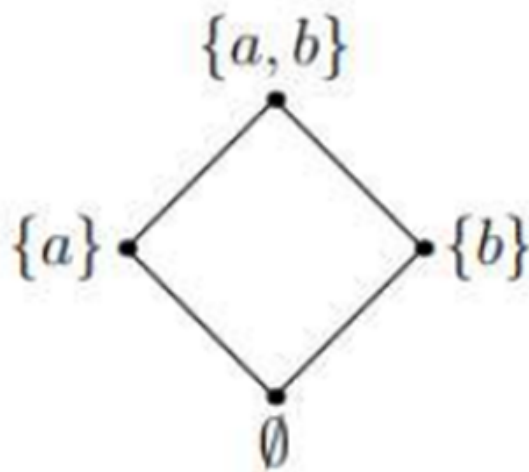


Figure: $A = \{a, b\}$.